CALCULATION OF SOME PROBLEMS OF NON-STANDARD FORM IN A CONVINENT WAY

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Abstract: *This article presents a convenient and easy way to calculate real equations with few unknowns. In general, it has an infinite number of solutions, but the most important task is to be able to estimate the unknown with a parameter.*

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Real equations with multiple unknowns have in general infinitely many solutions if they are solvable. In this case, an important task characterizing the set of solutions by using parameters. We are going to discuss two real equations and two parameterizations, but we will go beyond, showing how a simple idea can generate lots of nice problems, some of them really difficult.

Problem 1. Consider three real numbers a, b, c such that $abc = 1$ and write

$$x = a + \frac{1}{a}, y = b + \frac{1}{b}, z = c + \frac{1}{c} \quad (1)$$

Find an algebraic relation between x, y, z, independent of a, b, c. Of course, without any ideas, one would solve the equations from (1) with respect to *a, b, c* and then substitute the results in the relation $abc = 1$. But this is a mathematical crime! Here is a nice idea. To generate a relation involving x, y, z, we compute the product

$$xyz = \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) = \left(a^2 + \frac{1}{a^2}\right) + \left(b^2 + \frac{1}{b^2}\right)\left(c^2 + \frac{1}{c^2}\right) + 2$$
$$= (x^2 - 2) + (y^2 - 2) + (z^2 - 2) + 2$$

Thus,

$$x^2 + y^2 + z^2 - xyz = 4 \qquad (2)$$

and this is the answer to the problem. Now, another question appears: is the converse true? Obviously not (take for example the numbers $(x, y, z) = (1, 1, -1)$).

But looking again at (1), we see that we must have $min\{|x|, |y|, |z|\} \geq 2$. We will prove the following result.

Problem 2. Let *x, y, z* be real numbers with $max\{|x|, |y|, |z|\} > 2$. Prove that there exist real numbers a, b, c with $abc = 1$ satisfying (1). \Whenever we have a condition of the form $max\{|x|, |y|, |z|\} > 2$, it is better to make a choice. Here, let us take $|x| > 2$. This shows that there exists a nonzero real number u such that $x = u + \frac{1}{u}$, (we have used here the condition $|x| > 2$). Now, let us regard (2) as a second degree equation with respect to *z*. Since this equation has real roots, the discriminant must be nonnegative, which means that $(x^2 - 4)(y^2 - 4) \geq 0$. But since $|x| > 2$, we find that $y^2 \geq 4$ and so there exist a non-zero real number v for which $y = v + \frac{1}{v}$. How do we find the corresponding z? Simply by solving the second degree equation. We find two solutions:

$$z_1 = uv + \frac{1}{uv}, z_2 = \frac{u}{v} + \frac{v}{u}$$

and now we are almost done. If $z = uv + \frac{1}{uv}$ we take $(a, b, c) = \left(u, v, \frac{1}{uv}\right)$ and if $z = \frac{u}{v} + \frac{v}{u}$, then we take $(a, b, c) = \left(u, v, \frac{1}{uv}\right)$. All the conditions are satisfified and the problem is solved.

A direct consequence of the previous problem is the following:

If $x, y, z > 0$ are real numbers that verify (2), then there exist $\alpha, \beta, \chi \in R$ such that

$$x = 2ch(\alpha), y = 2ch(\beta), z = 2ch(\chi),$$

where $ch : R \to (0, \infty), ch(x) = \frac{e^x + e^{-x}}{2}$. Indeed, we write (1), in which this time it is clear that $a, b, c > 0$ and we take $\alpha = \ln a, \beta = \ln b, \chi = ln\, c$.

Inspired by the previous equation, let us consider another one

$$x^2 + y^2 + z^2 + xyz = 4, \quad (3)$$

where $x, y, z > 0$. We will prove that the set of solutions of this equation is the set of triples $(2\cos A, 2\cos B, 2\cos C)$ where A, B, C are the angles of an acute triangle. First, let us prove that all these triples are solutions. This reduces to the identity

$$\cos 2A + \cos 2B + \cos 2C + 2\cos A\cos B\cos C = 1.$$

This identity can be proved readily by using the sum-to-product formulas, but here is a nice proof employing geometry and linear algebra.

We know that in any triangle we have the relations

$$\begin{cases} a = c\cos B + b\cos C \\ b = a\cos C + c\cos A \\ c = b\cos A + a\cos B \end{cases}$$

which are simple consequences of the Law of Cosines. Now, let us consider the system

$$\begin{cases} x - y\cos C - z\cos B = 0 \\ -x\cos B + y\cos A = 0 \\ -x\cos B + y\cos A - z = 0 \end{cases}$$

From the above observation, it follows that this system has a nontrivial solution, that is (a, b, c) and so we must have

$$\begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix} = 0$$

which expanded gives

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

For the converse, we see first that $0 < x, y, z < 2$, hence there are numbers $A, B \in \left(0, \frac{\pi}{2}\right)$ such that $x = 2\cos A, y = 2\cos B$. Solving the equation with respect to z and taking into account that $z \in (0,2)$ we obtain $z = -2\cos(A + B)$. Thus we can take $C = \pi - A - B$ and we will have $(x, y, z) = (2\cos A, 2\cos B, 2\cos C)$. All in all we have solved the following problem.

Problem 3. The positive real numbers x, y, z satisfy (3) if and only if there exists an acute-angled triangle ABC such that

$x = 2\cos A, y = 2\cos B, z = 2\cos C$.

With the introduction and the easy problems over it is now time to see some nice applications of the above results.

Problem 4. Let $x, y, z > 2$ satisfying (2). We define the sequences $(a_n)_n \geq 1, (b_n)_n \geq 1, (c_n)_n \geq 1$ by

$$a_{n+1} = \frac{a_n^2 + x^2 - 4}{a_{n-1}}, b_{n+1} = \frac{b_n^2 + y^2 - 4}{b_{n-1}}, c_{n+1} = \frac{c_n^2 + z^2 - 4}{c_{n-1}},$$

with $a_1 = x, b_1 = y, c_1 = z$ and $a_2 = x^2 - 2, b_2 = y^2 - 2, c_2 = z^2 - 2$.

Prove that for all $n \geq 1$ the triple (a_n, b_n, c_n) also satisfies (2).

Solution. Let us write $x = a + \frac{1}{a}, y = b + \frac{1}{b}, z = c + \frac{1}{c}$ with $abc = 1$. Then

$$a_2 = a^2 + \frac{1}{a^2}, b_2 = b^2 + \frac{1}{b^2}, c_2 = c^2 + \frac{1}{c^2}$$

So, a reasonable conjecture is that

$$(a_n, b_n, c_n) = \left(a^n + \frac{1}{a^n}, b^n + \frac{1}{b^n}, c^n + \frac{1}{c^n}\right).$$

Indeed, this follows by induction from

$$\frac{\left(a^n + \frac{1}{a^n}\right)^2 + a^2 + \frac{1}{a^2} - 2}{a^{n-1} + \frac{1}{a^{n-1}}} = a^{n+1} + \frac{1}{a^{n+1}}$$

and two similar identities. We have established that

$$(a_n, b_n, c_n) = \left(a^n + \frac{1}{a^n}, b^n + \frac{1}{b^n}, c^n + \frac{1}{c^n}\right)$$

But if $abc = 1$, then certainly $a^n b^n c^n = 1$, which shows that indeed the triple (a_n, b_n, c_n) satisfies (2).

The following problem is a nice characterization of the equation (2) by polynomials and also teaches us some things about polynomials in two or three variables.

Problem 5. Find all polynomials $f(x, y, z)$ with real coeffiffifficients such that

$$f\left(a + \frac{1}{a}, b + \frac{1}{b}, c + \frac{1}{c}\right) = 0$$

whenever $abc = 1$.

Solution. From the introduction, it is now clear that the polynomials divisible by $x^2 + y^2 + z^2 - xyz - 4$ are solutions to the problem. But it is not obvious why any desired polynomial should be of this form. To show this, we use the classical polynomial long division. There are polynomials $g(x, y, z), h(y, z), k(y, z)$ with real coeffiffifficients such that

$$f(x,y,z) = (x^2+y^2+z^2-xyz-4)g(x,y,z) + xh(y,z) + k(y,z)$$

Using the hypothesis, we deduce that

$$0=\left(a+\frac{1}{a}\right)h\left(b+\frac{1}{b},c+\frac{1}{c}\right)+k\left(b+\frac{1}{b},c+\frac{1}{c}\right)$$

whenever $abc = 1$. Well, it seems that this is a dead end. Not exactly.

Now we take two numbers x, y such that $\min\{|x||y|\} > 2$ and we write

$$x=b+\frac{1}{b}, y=c+\frac{1}{c} \; with \; b=\frac{x+\sqrt{x^2-4}}{2}, c=\frac{y+\sqrt{y^2-4}}{2}$$

Then it is easy to compute $a+\frac{1}{a}$. It is exactly $xy+\sqrt{(x^2-4)(y^2-4)}$. So, we have found that

$$\left(xy+\sqrt{(x^2-4)(y^2-4)}\right)h(x,y)+k(x,y)=0$$

whenever $min\,\{|x||y|\} > 2$. And now? The last relation suggests that we should prove that for each y with $|y| > 2$, the function $x \to \sqrt{x^2-4}$ is not rational, that is, there aren't polynomials p, q such that $\sqrt{x^2-4}=\frac{p(x)}{q(x)}$. But this is easy because if such polynomials existed, than each zero of $x^2 - 4$ should have even multiplicity, which is not the case. Consequently, for each y with $|y| > 2$ we have $h(x,y) = k(x,y) = 0$ for all x. But this means that

$h(x,y) = k(x,y) = 0$ for all x, y, that is our polynomial is divisible with $x^2+y^2+z^2-xyz-4$.

O a different kind, the following problem and the featured solution prove that sometimes an efficient substitution can help more than ten complicated ideas.

Problem 6. Let $a, b, c > 0$. Find all triples (x,y,z) of positive real numbers such that

$$\begin{cases} x+y+z=a+b+c \\ a^2x+b^2y+c^2z+abc=4xyz \end{cases}$$

Solution. We try to use the information given by the second equation. This equation can be written as

$$\frac{a^2}{yz}+\frac{b^2}{zx}+\frac{c^2}{xy}+\frac{abc}{xyz}=4$$

and we already recognize the relation

$$u^2+v^2+\omega^2+uv\omega=4$$

where $u=\frac{a}{\sqrt{yz}}, v=\frac{b}{\sqrt{zx}}, \omega=\frac{c}{\sqrt{xy}}$. According to problem 3, we can find an acute-angled triangle *ABC* such that

$u = 2\cos A, v = 2\cos B, w = 2\cos C$.

We have made use of the second condition, so we use the first one to deduce that

$x + y + z = 2\sqrt{xy}\cos C + 2\sqrt{yz}\cos A + 2\sqrt{zx}\cos B$.

Trying to solve this as a second degree equation in $\sqrt{x}$, we find the discriminant

$-4(\sqrt{y}\sin C - \sqrt{z}\sin B)^2$.

Because this discriminant is nonnegative, we infer that

$\sqrt{y}\sin C = \sqrt{z}\sin B$ and $\sqrt{x} = \sqrt{y}\cos C + \sqrt{z}\cos B$.

Combining the last two relations, we find that

$$\frac{\sqrt{x}}{\sin A} = \frac{\sqrt{y}}{\sin B} = \frac{\sqrt{z}}{\sin C}$$

Now we square these relations and we use the fact that

$$\cos A = \frac{a}{2\sqrt{yz}}, \cos B = \frac{b}{2\sqrt{zx}} = \cos C = \frac{c}{2\sqrt{xy}}.$$

The conclusion is:

$$x = \frac{b+c}{2}, y = \frac{c+a}{2}, z = \frac{a+b}{2}$$

and it is immediate to see that this triple satisfies both conditions. Hence there is a unique triple that is solution to the given system.

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