R(SINX, COSX) KO'RINISHDAGI FUNKSIYALARNI INTEGRALLASH

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Annotatsiya: *Ushbu maqola o'qituvchi va o'quvchilarga metodik tavsiya sifatida yozilgan. Matematikaning asosiy bo'limlaridan biri integrallash haqida ma'lumot beriladi. O'quvchi bu R(sinx, cosx) misollarda o'rganish natijasida integrallash mavzusiga qiziqishi ortadi. Biz ushbu maqolada qiyin ikkita misolni yechimlarini keltrib otganmiz va shu mavzu yuzasidan misollar ham ko'rsatishga harakat qildik. integrallash oid dars jarayonida o'qituvchi maqoladan ko'rgazma sifatida foydalansa bo'ladi. Maqola matematikani o'qitish samaradorligini oshirishda xizmat qiladi.Bu maqolamiz sizlarga manzur bo'ladi degan umiddamiz.*

Kalit so'zlar: *trigonometrik funksiyalar, misol, yechish, isbotlash, darajani pasaytirish formulalar.*

Hamma trigonometrik funksiyalarni sinx va cosx orqali ratsional ko'rinishda ifodalash mumkin.

Bu ifodani R (sinx,cosx) orqali belgilaymiz.

Endi R (sinx,cosx) ko'rinishdagi ifodani integrallash kerak bo'lsin.

$$\int R(\sin x, \cos x)dx$$

Bunday integralni $tg\frac{x}{2} = z$ belgilash yordamida z o'zgaruvchili ratsional funksiyaning integraliga almashtirish mumkin. Integralni bunday almashtirish ratsionallashtirish deyiladi. Haqiqatdan ham, $tg\frac{x}{2} = z$ desak,

$$\sin x = \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2z}{1+z^2};$$

$$\cos x = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1 - z^2}{1 + z^2};$$

$$\frac{x}{2} = arctgz$$

$$x = 2arctgz$$

$$dx = \frac{2dz}{1 + z^2}$$

Shuning uchun

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2z}{1 + z^2}; \frac{1 - z^2}{1 + z^2}\right) \cdot \frac{2dz}{1 + z^2} = \int R_1(z)dz$$

bunda $R_1(z)$-z o'zgaruvchili ratsional funksiya.

Bunday almashtirish R(sinx,cosx) ko'rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun bunday almashtirish universal trigonometrik almashtirish deyiladi. Lekin bunday almashtirish ko'pincha ancha murakkab ratsional funksiyaga olib keladi. Shuning uchun, sodda o'rniga qo'yishlardan ham foydalansa bo'ladi. Masalan:

1) Agar R(sinx,cosx) funksiya sinx ga nisbatan toq bo'lsa, ya`ni

R(-sinx,cosx)-R(sinx,cosx) bo'lsa, u holda z=cosx; dz=-sinxdx o'rniga qo'yish bu funksiyani ratsionallashtiradi.

2) Agar R(sinx,cosx) funksiya cosx ga nisbatan toq bo'lsa, ya`ni

R(sinx,-cosx)-R(sinx,cosx)bo'lsa, u holda z=sinx; dz=cosxdx o'rniga qo'yish bu funksiyani ratsionallashtiradi.

3) Agar R(sinx,cosx) funksiya sinx va cosx ga nisbatan juft bo'lsa, ya`ni R(-sinx,-cosx)=R(sinx,cosx) bo'lsa, u Holda $z = tg; dx = \frac{dz}{1 + z^2}; x = arctgz$ o'rniga qo'yish bu funksiyani ratsionallashtiradi. Bu holda

$$Sin^2 x = \frac{tg^2 x}{1 + tg^2 x} = \frac{z^2}{1 + z^2};$$

$$\cos^2 x = \frac{1}{1 + tg^2 x} = \frac{1}{1 + z^2};$$

bo'ladi.

1-Misol $J = \int \frac{dx}{4\sin x\_ + 3\cos x + 5}$ integralni hisoblang.

Yechish: $tg\frac{x}{2} = z$ o'rniga qo'yishdan foydalanamiz.

$$\sin x = \frac{2z}{1+z^2};\ \cos x = \frac{1-z^2}{1+z^2};\ dx = \frac{2dz}{1+z^2}$$

$$J = \int \frac{\frac{2dz}{1+z^2}}{4\cdot\frac{2z}{1+z^2} + 3\cdot\frac{1-z^2}{1+z^2} + 5} = \int \frac{2dz}{(1+z^2)\cdot\frac{8z+3-3z^2+5+5z^2}{1+z^2}} = \int \frac{2dz}{2z^2+8z+8} =$$

$$= \int \frac{2dz}{2(z^2+4z+4)} = \int \frac{dz}{(z+2)^2} = -\frac{1}{z+2} + C = -\frac{1}{tg\frac{x}{2}+2} + C;$$

2-Misol. $J = \int \frac{dx}{1+\sin^2 x}$ integralni hisoblang.

Yechish :

Integral belgisi ostidagi funksiya juft funksiya, shuning uchun tgx=z almashtirishni bajaramiz.

U holda z=tgx; x=arctgz; $dx = \frac{dz}{1+z^2}$ $\sin^2 x = \frac{z^2}{1+z^2}$;

Natijada quyidagini hosil qilamiz:

$$J = \int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{dt}{1+z^2}}{1+\frac{z^2}{1+z^2}} = \int \frac{\frac{dz}{1+z^2}}{\frac{1+z^2+z^2}{1+z^2}} = \int \frac{dz}{1+2z^2} = \frac{1}{2}\int \frac{dz}{\frac{1}{2}+z^2} = \frac{1}{2}\int \frac{dz}{\left(\sqrt{\frac{1}{2}}\right)^2 + z^2} =$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}} arctg\frac{z}{\sqrt{\frac{1}{2}}} + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}\,Z + C = \frac{\sqrt{2}}{2} arctg\sqrt{2}tgx + C;$$

3-Misol: $J = \int \frac{\sin^3 x}{2+\cos x} dx$ integralni hisoblang.

Yechish: Integral ostidagi funksiya sinx ga nisbatan toq funksiya . Shuning uchun z=cosx; dz=-sinxdx⇒ sinxdx=-dz almashtirishni bajaramiz:

$$J = \int \frac{\sin^2 x\cdot\sin x dx}{2+\cos x} = \int \frac{(1-\cos^2 x)\sin x dx}{2+\cos x} = -\int \frac{(1-z^2)dz}{2+z} = \int \frac{z^2-1}{2+z}dz = \int\left(z-2+\frac{3}{z+2}\right)dz =$$

$$= \frac{z^2}{2} - 2z + 3\ln|z+2| + C = \frac{\cos^2 x}{2} - 2\cos x + 3\ln|\cos x + 2| + C;$$

4) Agar R(sinx, cosx) funksiya sinx va cosx darajalarining ko'paytmasi bo'lsa, ya`ni $\int \sin^n x\cdot\cos^m x dx$ ko'rinishdagi integralni hisoblash, m va n ga bog'liq holda turli o'rniga qo'yishlar bajariladi:

a) Agar n>0 va toq bo'lsa, u holda cosx=z; sinxdx=-dz o'rniga qo'yish integralni ratsionallashtiradi.

b) Agar m>0 va toq bo'lsa, u holda sinx=z; cosxdx=dz o'rniga qo'yish bajariladi.

4-Misol: $J=\int\frac{\sin^3 x}{\cos^4 x}dx$ integralni hisoblang.

Yechish: cosx=z; sinxdx=-dz almashtirishni bajaramiz:

$$J=\int\frac{\sin^2 x\sin x}{\cos^4 x}dx=\int\frac{(1-\cos^2 x)\sin x dx}{\cos^4 x}=-\int\frac{(1-z)^2dz}{z^4}=-\int\frac{dz}{z^4}+\int\frac{z^2}{z^4}dz=\frac{1}{3z^3}-\frac{1}{z}+C=$$
$$\frac{1}{3\cos^3 x}-\frac{1}{\cos x}+C;$$

v) Agar ikkala n va m ko'rsatkichlar juft va nomanfiy bo'lsa, u holda trigonometriyadan ma`lum bo'lgan

$$\sin^2 x=\frac{1-\cos 2x}{2};\quad \cos^2 x=\frac{1+\cos 2x}{2}$$

darajani pasaytirish formulalaridan foydalanamiz.

5-Misol. $J=\int\sin^4 xdx$ integralni hisoblang.

Yechish: Darajani pasaytirish formulasidan foydalanamiz.

$$J=\int\sin^4 xdxd=\int(\sin^2 x)^2dx=\int\left(\frac{1-\cos 2x}{2}\right)^2dx=\frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx=$$
$$=\frac{1}{4}\int\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx=\frac{1}{4}x-\frac{1}{2}\frac{\sin 2x}{2}+\frac{1}{8}x+\frac{1}{8}\frac{\sin 4x}{4}+C=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}$$
$$\sin 4x+C;$$

g) Agar m+n=-2K≤0 (juft, nomusbat) bo'lsa, u holda tgx=z yoki z=ctgx o'rniga qo'yish integralni darajali funksiyalarning integrallari yig'indisiga olib keladi.

6-Misol. $J=\int\frac{dx}{\sin^3 x\cdot\cos x}$ integralni hisoblang.

Yechish: bu yerda n=-3; m=-1; m+n=-4<0

$$J=\int\frac{dx}{\sin^3 x\cdot\cos x}=\int\frac{\sin^2 x+\cos^2 x}{\sin^3 x\cdot\cos x}dx=\int\frac{1}{\sin x\cdot\cos x}dx+\int\frac{\cos x}{\sin^3 x}dx=2\int\frac{dx}{\sin 2x}+\int\frac{d(\sin x)}{\sin^3 x}=$$
$$=\ln|tgx|-\frac{1}{2\sin^2 x}+C;$$

7-Misol . $J=\int\frac{\sin^2 x}{\cdot\cos^6 x}dx$ integralni hisoblang.

Yechish: bu yerda n=2, m=-6; n+m=-4<0 quyidagini almashtirishni bajaramiz.

z=tgx; x=arctgz; dx= $\frac{dz}{1+z^2}$

$$\frac{\sin^2 x}{\cos^6 x}=\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^4 x}=tg^2x\left(\frac{1}{\cos^4 x}\right)=tg^2x(1+tg^2x)^2=z^2(1+z^2)^2,$$

Natijada quyidagini hosil qilamiz.

$$J=\int\frac{\sin^2 x}{\cos^6 x}dx=\int z^2(1+z^2)\frac{dz}{1+z^2}=\int z^2+z^4)dz=\int z^2dz+\int z^4dz=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C;$$

8-Misol. $J=\int ctg^4 x dx$ integralni hisoblang.

Yechish: bu yerda $ctg^4 x=\frac{\cos^4 x}{\sin^4 x}$ desak, m=4; n=-4; m+n=0;

Quyidagi almashtirishni bajaramiz.

ctgx=z; x=arcctgz; $dx=-\frac{dz}{1+z^2}$

Natijada

$$J=\int ctg^4 x dx=-\int z^4\cdot\frac{dz}{1+z^2}=-\int\left(z^2-1+\frac{1}{1+z^2}\right)dz=\int dz-\int z^2dz-\int\frac{dz}{1+z^2}=z-\frac{z^3}{3}+arctgz+C=$$

$$=ctgx-\frac{1}{3}ctg^3x+arctgctgx+C$$

9-Misol. $J=\int\frac{dx}{\cos^6 x}$ integralni hisoblang.

Yechish : bu yerda n=0 ; m=-6; m+n=-6<0 quyidagi almashtirishni bajaramiz.

$$tg=x;\qquad \cos^2 x=\frac{1}{1+z^2};\qquad \frac{dx}{\cos^2 x}=dz;$$

U holda

$$J=\int\frac{dx}{\cos^6 x}=\int\frac{1}{\cos^4 x}\cdot\frac{1}{\cos^2 x}dx=\int\frac{dz}{\left(\frac{1}{1+z^2}\right)}=\int(1+z^2)^2dz=\int(1+2z^2+z^4)dz=z+\frac{2}{3}z^3+\frac{z^5}{5}+C$$

$$=tgx+\frac{2}{3}tg^3x+\frac{1}{5}tg^5x+C$$

d) Agar darajalardan biri nolga teng, ikkinchisi manfiy toq son bo'lsa, u holda $tg\frac{x}{2}=z$ almashtirish bajariladi.

10-Misol. $J=\int\frac{dx}{\sin^3 x}$ integralni hisoblang.

Yechish: Quyidagicha almashtirishni bajaramiz.

$$tg\frac{x}{2}=z;\qquad dx=\frac{2dz}{1+z^2};\qquad \sin x=\frac{2z}{1+z^2}$$

Natijada:

$$J=\int\frac{dx}{\sin^3 x}=\int\frac{dx}{\sin^2 x\cdot\sin x}=\int\frac{\frac{2dz}{1+z^2}}{\left(\frac{2z}{1+z^2}\right)^2\cdot\frac{2z}{1+z^2}}=\int\frac{2\cdot(1+z^2)^3}{(1+z^2)\cdot 4z^2\cdot 2z}dz=\frac{1}{4}\int\frac{(1+z^2)^2}{z^3}dz=$$

$$=\frac{1}{4}\int\frac{1+2z^2+z^4}{z^3}dz=\frac{1}{4}\int\left(\frac{1}{z^3}+\frac{2}{z}+z\right)dz=-\frac{1}{8z^2}+\frac{1}{2}\ln|z|+\frac{1}{4}\cdot\frac{z^2}{2}+C=-\frac{1}{8}ctg^2\frac{x}{2}+\frac{1}{2}\ln\left|tg\frac{x}{2}\right|+$$

$$+\frac{1}{8}tg^2\frac{x}{2}+C;$$

5) Quyidagi ko'rinishdagi integrallarni qarab chiqamiz.

$$\int \cos nx \cdot \cos mx dx$$

$$\int \sin nx \cdot \cos mx dx$$

$$\int \sin nx \cdot \sin mx dx$$

Bularni integrallash uchun trigonometrik funksiyalarning ko'paytmasini yig'indiga almashtiruvchi formulalar yordamida olinadi:

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

11-Misol . $J = \int \sin 3x \cdot \cos 2x dx$ integralni hisoblang.

Yechish: Integral ostidagi ko'paytmani yig'indiga almashtiramiz.

$$J = \int \sin 3x \cdot \cos 2x dx = \frac{1}{2}\int (\sin 5x + \sin x) dx = -\frac{1}{2} \cdot \frac{\cos 5x}{5} - \frac{1}{2} \cdot \cos x + C = -\frac{1}{10} \cdot \frac{\cos 5x}{1} - \frac{1}{2} \cdot \cos x + C$$