METHOD OF UNDETERMINED COEFFICIENTS

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Annotatsiya: *Ma'lumki funksiyalarning maksimum qiymatini topish masalasi, biz odatlanib qolgan usullar orqali birmuncha qiyinchiliklarga olib keladi.Ba'zida funksiya grafigi va uning hosilasi yordamida maksimum qiymatlarni aniqlash oson bo'lmasligi mumkin.Biz ushbu maqolada turli xil ko'rinishlardagi bir o'zgaruvchili va ko'p o'zgaruvchili funksiyalarning maksimumini hisoblashning aniqlanmagan koefisiyentlar usulini bayon qilamiz.*

Kalit so'zlar: *koshi tengsizligi, maksimum qiymat, almashtirishlar*

Abstract: *It is known that the problem of finding the maximum value of functions leads to some difficulties through the methods we are used to. Sometimes it may not be easy to determine the maximum values using the graph of the function and its derivative. we describe the undefined coefficients method for calculating the maximum of multivariable functions.*

Keywords: *Cauchy inequality, maximum value, permutations.*

INTRODUCTION

The problem of finding the maximum values of functions is a rather urgent problem. In this article, we deal with the problem of finding the maximum value of various types of functions using the method of undefined coefficients.

Problem 1. Assume x, y, z are real numbers that are not all 0. Find the maximum value for $\frac{xy+2yz}{x^2+y^2+z^2}$.

Hint. To find the maximum for $\frac{xy+2yz}{x^2+y^2+z^2}$, we only need to show that there exists a constant c, such that

$$\frac{xy+2yz}{x^2+y^2+z^2} \leq c \qquad (1.1)$$

And that the equal sign holds for some x, y, z.

(1.1) can be translated to $x^2 + y^2 + z^2 \geq \frac{1}{c}(xy + 2yz)$.

Since the right hand side has terms xy and $2yz$, we split the trem y^2 in the left hand into δy^2 and $(1-\delta)y^2$.

Since,

$$x^2 + \delta y^2 \geq 2\sqrt{\delta}\, xy$$

$$z^2 + (1-\delta)y^2 \geq 2\sqrt{1-\delta}\, yz$$

We want $\frac{\sqrt{1-\delta}}{\sqrt{\delta}} = 2$, which gives $\delta = \frac{1}{5}$.

Now, putting the value of the parameter δ defined above into the required expressions, we get the following

Since,

$$x^2 + \frac{1}{5}y^2 \geq \frac{2}{\sqrt{5}}\,xy$$

$$z^2 + \frac{4}{5}y^2 \geq \frac{4}{\sqrt{5}}\,yz$$

We obtain

$$x^2 + y^2 + z^2 \geq \frac{2}{\sqrt{5}}(xy + 2yz).$$

or $\frac{xy+2yz}{x^2+y^2+z^2} \leq \frac{\sqrt{5}}{2}$

the equality holds when $x = 1, y = \sqrt{5}\ , z = 2$

Hence, the maximum $\frac{\sqrt{5}}{2}$ can be reached.

Now let's consider an excellent application of the method of indefined coefficients in finding the maximum value of the following function of one variable given on a segment.

Problem 2. For $\frac{1}{2} \leq x \leq 1$, find the maximum value of

$$(1+x)^5(1-x)(1-2x)^2$$

Solution: Let us consider the maximum value of

$$(\alpha(1+x))^5(\beta(1-x))(\gamma(1-2x))^2$$

Where α, β, γ are positive integers satisfying $5\alpha - \beta + 4\gamma = 0,$

$$(\alpha(1+x) = \beta(1-x) = \gamma(2x-1)$$

This implies

$\frac{\beta-\alpha}{\beta+\alpha} = \frac{\beta+\gamma}{\beta+2\gamma}$ and plugging in $\beta = 5\alpha + 4\gamma$

We have

$$0 = 2(3\alpha\gamma + 5\alpha^2 - 2\gamma^2) = 2(5\alpha - 2\gamma)(\alpha + \gamma)$$

Let $(\alpha, \beta, \gamma) = (2,30,5)$ and from AM-GM inequality, we obtain

$$(2(1+x))^5(30(1-x))(5(1-2x))^2 \leq (\frac{15}{4})^8$$

The equality is achieved when $x = \frac{7}{8}$.

As a result, the maximum value for

$(1+x)^5(1-x)(1-2x)^2$ is $\frac{3^7 * 5^5}{2^{22}}$.

Now we give the complete proof of the problem proposed by Ostrowski below.

Problem 3. (Ostrowski) Assume two sets of real numbers $a_1, a_2, a_3, a_4 \ldots\ldots a_{n-1}, a_n$ and $b_1, b_2, b_3, b_4 \ldots\ldots b_{n-1}, b_n$ are not scaled version of each other.

Real numbers $x_1, x_2, x_3, x_4 \ldots\ldots x_{n-1}, x_n$ satify:

$a_1x_1 + a_2x_2 + \ldots. + a_nx_n = 0$

$b_1x_1 + b_2x_2 + \ldots. + b_nx_n = 1$

Prove that

$x_1^2 + x_2^2 + \cdots \ldots + x_n^2 \geq \frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i^2 * \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2}$.

Proof: Assume $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2 + \alpha \sum_{i=1}^n a_i\, x_i + \beta\left(\sum_{i=1}^n b_i\, x_i - 1\right)$

where α, β are undetermined coefficients.

Thus,

$$\sum_{i=1}^n x_i^2 = \sum_{i=1}^n (x_i + \frac{\alpha a_i + \beta b_i}{2})^2 - \sum_{i=1}^n \left(\frac{\alpha a_i + \beta b_i}{2}\right)^2 - \beta \geq \; -\sum_{i=1}^n \left(\frac{\alpha a_i + \beta b_i}{2}\right)^2 - \beta$$

For the above inequality, the equal sign holds if only if

$x_i = -\frac{\alpha a_i + \beta b_i}{2} \quad (i = 1,2,3, \ldots. \;, n) \qquad (*)$

Substitute (*) back into $a_1 x_1 + a_2 x_2 + \ldots. + a_n x_n = 0$ and $b_1 x_1 + b_2 x_2 + \ldots. + b_n x_n = 1$, we have

$$-\frac{1}{2}\alpha A - \frac{1}{2}\beta C = 0$$
$$-\frac{1}{2}\alpha C - \frac{1}{2}\beta B = 1$$

Here, $A = \sum_{i=1}^n a_i^2$, $B = \sum_{i=1}^n b_i^2$, $C = \sum_{i=1}^n a_i\, b_i$

Therefore,

$$\alpha = \frac{2C}{AB - C^2}\,,\quad \beta = -\frac{2A}{AB - C^2}$$

Hence

$$\sum_{i=1}^n x_i^2 = -\sum_{i=1}^n \left(\frac{\alpha a_i + \beta b_i}{2}\right)^2 - \beta = \frac{A}{AB - C^2}$$

Given a function with n variables, let's look at another great way to find the maximum value of a given function.

Problem 4. Find the maximum M_n for the function

$f_n(x_1, x_2, \ldots.., x_n) = \frac{x_1}{(1+x_1+\cdots..+x_n)^2} + \frac{x_2}{(1+x_2+\cdots..+x_n)^2} + \cdots.. + \frac{x_n}{(1.+x_n)^2}$

where $x_i \geq 0$

Moreover, express M_n using M_{n-1} and find $\lim_{n\to\infty} M_n$

Solution:

Let $a_i = \frac{1}{1+x_i+\cdots\ldots+x_n}$, $1 \leq i \leq n$. Define $a_{n+1} = 1$.

Thus,

$$1 + x_i + x_{i+1} + \cdots \ldots + x_n = \frac{1}{a_i}$$

Also, $1 + x_{i+1} + x_{i+2} + \cdots \ldots + x_n = \frac{1}{a_{i+1}}$

Hence, $x_i = \frac{1}{a_i} - \frac{1}{a_{i+1}}$

Substituting $x_i's$, we have

$$f_n = \sum_{i=1}^{n} a_i^2 \left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right) = \sum_{i=1}^{n} (a_i - \frac{a_i^2}{a_{i+1}})$$
$$= (a_1 + a_2 + \cdots .. + a_n) - (\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots .. + \frac{a_n^2}{1})$$

To find the maximum for f_n, we construct the following inequalities

$$\frac{a_1^2}{a_2} + \varphi_1^2 a_2 \geq 2\varphi_1 a_1$$

$\frac{a_2^2}{a_3} + \varphi_2^2 a_3 \geq 2\varphi_2 a_2$ (1)

....

$$\frac{a_n^2}{1} + \varphi_n^2 \geq 2\varphi_n a_n$$

Here $\varphi_1, \varphi_2 \dots \dots, \varphi_n$ are parameters, $\varphi_i \geq 0$.
Adding (1), if we let

$$2\varphi_1 = 1$$

$2\varphi_2 = 1 + \varphi_1^2$ (2)

....

$$2\varphi_n = 1 + \varphi_{n-1}^2$$

Then immediately $f_n \leq \varphi_n^2$.
Note that $\varphi_i \geq \varphi_{i-1}$ and $0 \leq \varphi_i \leq 1$. Therefore, $\lim_{n\to\infty} \varphi_n$ exists.
It is easy to see that the limit is 1.

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