BA`ZI BIR ELEMENTAR FUNKSIYALAR UCHUN MAKLOREN FORMULASI

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Annotatsiya: *Ushbu maqolada Matematika fanining asosiy yordamchisi hisoblanadi. Bu maqolani avzali tomonlari shundaki koshi teoremasi, koshining integral formulasi va Ba`zi bir elementar funksiyalar uchun Makloren formulasi keltirib o`tilgan. Ba`zi bir elementar funksiyalar uchun Makloren formulasi istalgan tartibli hosilaga ega bo'lishi haqida yangi elementlarga ega bo`lishga ushbu maqola yordam beradi.*

Kalit so'zlar: *Makloren formulasi, e^x funksiya, Sinus funksiya uchun Makloren formulasi, Teylor formulasi yordamida taqribiy hisoblash, Makloren formulasi Lagranj ko`rinishdagi qoldiq hadi, $R_n(x)$ Teylor formulasi.*

e^x funksiya uchun Makloren formulasi. $f(x)=e^x$ funksiyaning $(-\infty;+\infty)$ oraliqda barcha tartibli hosilalari mavjud: $f^{(k)}(x)=e^x$, k=1, 2, ..., n+1. Bundan x=0 da $f^{(k)}(0)=1$, k=1, 2, ..., n; $f^{(n+1)}(\theta x)=e^{\theta x}$ va $f(0)=1$ hosil bo`ladi. Olingan natijalarni formulaga qo`yib

$$\mathring{a}^{\tilde{o}} = 1+\frac{\tilde{o}}{1!}+\frac{\tilde{o}^2}{2!}+\ldots+\frac{\tilde{o}^n}{n!}+\frac{\tilde{o}^{n+1}}{(n+1)!}e^{\theta x}$$

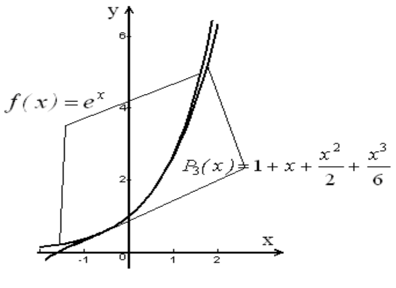
bu yerda $0<\theta<1$, formulaga ega bo`lamiz.

1-rasmda $f(x)=e^x$ funksiya va $P_3(x)$ ko`phad funksiyaning grafiklari keltirilgan.

Agar x=1 bo`lsa,

$$\mathring{a} = 1+\frac{1}{1!}+\frac{2}{2!}+\ldots+\frac{1}{n!}+\frac{\mathring{a}^{\theta}}{(n+1)!}$$

formulaga ega bo`lamiz. Bu formula yordamida e soning irratsionalligini isbot qilish mumkin.



1-rasm

Haqiqatan ham, faraz qilaylik, $\mathring{a}=\frac{p}{q}$ - ratsional son bo`lsin. Bunda e>1 bo`lganligi uchun p>q bo`ladi. (4.2) da $\mathring{a}=\frac{p}{q}$ desak,

$$\frac{\mathrm{p}}{\mathrm{q}}=2+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{n!}+\frac{1}{(n+1)!}\left(\frac{\mathrm{p}}{\mathrm{q}}\right)^{\theta}$$

Bu tenglikning ikkala tomonini n! ga ko`paytirsak quyidagi tenglikni hosil qilamiz:

$$\frac{\mathrm{p}}{\mathrm{q}}n!-(2\cdot n!+\frac{1}{2!}\cdot n!+\frac{1}{3!}\cdot n!+...+1)=\frac{1}{n+1}\left(\frac{\mathrm{p}}{\mathrm{q}}\right)^{\theta}$$

Bu yerda n sonni r dan katta deb olishimiz mumkin. U holda θ<1, p>q bo`lganligi uchun

$$0<\frac{1}{n+1}\left(\frac{\mathrm{p}}{\mathrm{q}}\right)^{\theta}<\frac{1}{n+1}\frac{\mathrm{p}}{\mathrm{q}}\leq\frac{\mathrm{p}}{\mathrm{n}+1}<1$$

bo`ladi. Shuningdek, n>p>q bo`lganligi uchun $0<\frac{1}{n+1}\left(\frac{\mathrm{p}}{\mathrm{q}}\right)^{\theta}<\frac{1}{n+1}\frac{\mathrm{p}}{\mathrm{q}}\leq\frac{\mathrm{p}}{\mathrm{n}+1}<1\mathrm{n}!$ - butun son, chunki n! da q ga teng bo`lgan ko`paytuvchi uchraydi.

Ravshanki,

$$2n!+\frac{1}{2!}\cdot n!+\frac{1}{3!}\cdot n!+...+1$$

ko`rinishdagi yig`indi ham butun son bo`ladi. Demak, n>p uchun (4.3) tenglikning chap tomoni musbat butun son, o`ng tomoni esa (4.4) ga ko`ra birdan kichik musbat son bo`ladi. Bu kelib chiqqan ziddiyat e sonining ratsional son deb faraz qilishimizning noto`g`ri ekanligini ko`rsatadi. Shuning uchun e – irratsional son bo`ladi.

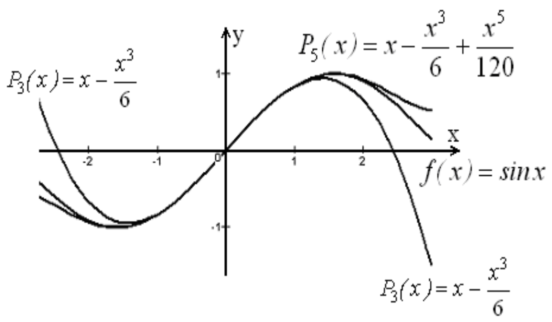
Sinus funksiya uchun Makloren formulasi. f(x)=sinx funksiyaning istalgan tartibli hosilasi mavjud va n-tartibli hosila uchun quyidagi formula o`rinli edi (I.8-§): $f^{(n)}(x)=\sin(x+\frac{n\pi}{2})$. x=0 da f(0)=0 va

$$f^{(n)}(0)=\sin\frac{n\pi}{2}=\begin{cases}0, & agar \quad n=2k,\\ (-1)^{\hat{e}}, & agar \quad n=2\hat{e}+1\end{cases}$$

Shuning uchun (3.10) formulaga ko`ra

$$\sin x=x-\frac{x^3}{3!}+...+(-1)^k\frac{x^{2k+1}}{(2k+1)!}+\frac{x^{2k+2}}{(2k+2)!}\sin(\theta x+(k+1)\pi),\quad 0<\theta<1$$

ko`rinishdagi yoyilmaga ega bo`lamiz.



2-rasm

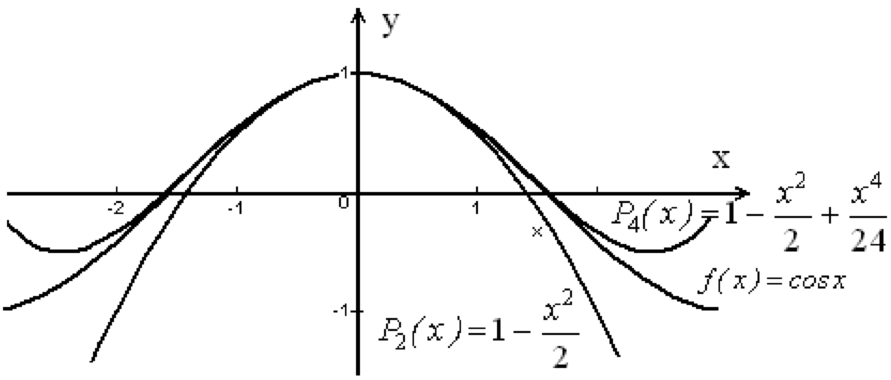
2-rasmda f(x)=sinx, $P_3(x)$, $P_5(x)$ funksiyalarning grafiklari keltirilgan.

Kosinus funksiya uchun Makloren formulasi. Ma`lumki, f(x)=cosx funksiyaning n-tartibli hosilasi uchun $f^{(n)}(x)=\cos(x+\frac{n\pi}{2})$ formulaga egamiz.

x=0 da f(0)=1 va $f^{(n)}(0)=\cos\frac{n\pi}{2}=\begin{cases}0, & agar \quad n=2k+1,\\ (-1)^k, & agar \quad n=2k\end{cases}$

Demak, cosx funksiya uchun quyidagi formula o`rinli:

$$\tilde{n}osx=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...+(-1)^k\frac{x^{2k}}{2k!}+\frac{x^{2k+2}}{(2k+1)!}\cos(\theta x+k\pi+\frac{\pi}{2}),\quad 0<\theta<1 \quad (4.6)$$



3-rasm

3-rasmda f(x)=cosx, $P_2(x)$, $P_4(x)$ funksiyalarning grafiklari keltirilgan.

f(x)=$(1+x)^{\mu}$ ($\mu\in R$) funksiya uchun Makloren formulasi. Bu funksiya (-1;1) intervalda aniqlangan va cheksiz marta differensiallanuvchi. Uni Makloren formulasiga yoyish uchun f(x)=$(1+x)^{\mu}$ funksiyadan ketma-ket hosilalar olamiz:

$f'(x)=\mu(1+\tilde{o})^{\mu-1}$, $\quad f''(x)=\mu(\mu-1)(1+x)^{\mu-2}$,

$f'''(x)=\mu(\mu-1)(\mu-2)(1+x)^{\mu-3},...$,

$$f^{(n)}(x)=\mu(\mu-1)...(\mu-n+1)(1+x)^{\mu-n}. \qquad (4.7)$$

Ravshanki, f(0)=1, $f^{(n)}(0)=\mu(\mu-1)...(\mu-n+1)$. Shuning uchun f(x)=$(1+x)^{\mu}$ funksiyaning Makloren formulasi quyidagicha yoziladi:

$$(1+x)^{\mu}=1+\mu x+\frac{\mu(\mu-1)}{2!}x^2+...+\frac{\mu(\mu-1)...(\mu-n+1)}{n!}x^n+\frac{\mu(\mu-1)...(\mu-n)}{(n+1)!}(1+\theta x)^{\mu-n-1}x^{n+1}$$
(4.8)

$0<\theta<1$.

$f(x)=\ln(1+x)$ funksiya uchun Makloren formulasi. Bu funksiyaning $(-1;\infty)$ intervalda aniqlangan va istalgan tartibli hosilasi mavjud. Haqiqatan ham, $f'(x)=(\ln(1+õ))'=(1+x)^{-1}$ funksiyasiga (4.7) formulani qo`llab, unda $\mu=-1$ deb n ni n-1 bilan almashtirsak, $f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ formulani hosil qilamiz. Ravshanki, $f(0)=0$, $f^{(n)}(0)=(-1)^{n-1}(n-1)!$ Shuni e`tiborga olib, berilgan funksiyaning Makloren formulasini yozamiz:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...+(-1)^{n-1}\frac{x^n}{n}+\frac{(-1)^n}{(n+1)}\frac{x^{n+1}}{(1+\theta x)^{n+1}},\quad 0<\theta<1 \qquad (4.9)$$

Yuqorida keltirilgan asosiy elementar funksiyalarning Makloren formulalari boshqa funksiyalarni Teylor formulasiga yoyishda foydalaniladi. Shunga doir misollar ko`ramiz.

Misol. Ushbu $f(x)=e^{-3x}$ funksiya uchun Makloren formulasini yozing.

Yechish. Bu funksiyaning Makloren formulasini yozish uchun $f(0)$, $f`(0)$,...,$f^{(n)}(0)$ larni topib, (3.10) formuladan foydalanish mumkin edi. Lekin $f(x)=e^x$ funksiyaning yoyilmasidan foydalanish ham mumkin. Buning uchun (4.1) formuladagi x ni -3x ga almashtiramiz, natijada

$$å^{-3õ}=1-\frac{3õ}{1!}+\frac{9õ^2}{2!}-...+(-1)^n\frac{3^n õ^n}{n!}+\frac{(-3õ)^{n+1}}{(n+1)!}e^{-3\theta x},\ 0<\theta<1,$$

formulaga ega bo`lamiz.

Misol. Ushbu $f(x)=\ln x$ funksiyani $x_0=1$ nuqta atrofida Teylor formulasini yozing.

Yechish. Berilgan funksiyani Teylor formulasiga yoyish uchun $f(x)=\ln(1+x)$ funksiya uchun olingan (4.9) asosiy yoyilmadan foydalanamiz. Unda x ni x-1 ga almashtiramiz, natijada $\ln x=\ln((x-1)+1)$ va

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+...+(-1)^{n-1}\frac{(x-1)^n}{n}+\frac{(-1)^n}{(n+1)}\cdot\frac{(x-1)^{n+1}}{(1+\theta(x-1))^{n+1}},\ 0<\theta<1$$

formulaga ega bo`lamiz. Bu formula $x-1>-1$ bo`lganda, ya`ni $x>0$ larda o`rinli.

Teylor formulasi yordamida taqribiy hisoblash. Makloren formulasi Lagranj ko`rinishdagi qoldiq hadini baholash masalasini qaraylik.

Faraz qilaylik, shunday o`zgarmas M son mavjud bo`lsinki, argument x ning $x_0=0$ nuqta atrofidagi barcha qiymatlarida hamda n ning barcha qiymatlarida $|f^{(n)}(x)|\le M$ tengsizlik o`rinli bo`lsin. U holda

$$|R_n(x)|=\left|(x-1)-\frac{(x-1)^2}{2}+...+(-1)^{n-1}\frac{(x-1)^n}{n}+\frac{(-1)^n}{(n+1)}\cdot\frac{(x-1)^{n+1}}{(1+\theta(x-1))^{n+1}}\right|\le M\cdot\frac{|x|^{n+1}}{(n+1)!}$$

tengsizlik o`rinli bo`ladi. Argument x ning tayin qiymatida $\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}=0$ tenglik o`rinli, demak n ning yetarlicha katta qiymatlarida $R_n(x)$ yetarlicha kichik bo`lar ekan.

Shunday qilib, $x_0=0$ nuqta atrofida f(x) funksiyani

$f(0)+f`(0)x+\frac{1}{2!}f``(0)x^2+\ldots+\frac{1}{n!}f^{(n)}(0)x^n$

ko`phad bilan almashtirish mumkin. Natijada funksiyaning x nuqtadagi qiymati uchun

$f(x)\approx f(0)+f`(0)x+\frac{1}{2!}f``(0)x^2+\ldots+\frac{1}{n!}f^{(n)}(0)x^n$

taqribiy formula kelib chiqadi. Bu formula yordamida bajarilgan taqribiy hisoblashdagi xatolik $|R_n(x)|$ ga teng bo`ladi.

Misol. $e^{0,1}$ ni 0,001 aniqlikda hisoblang.

Yechish. e^x funksiyaning Makloren formulasidan foydalanamiz. (4.1) formulada x=0,01 deb olsak, u holda

$$\mathring{a}^{0,1}\approx 1+\frac{0,1}{1!}+\frac{0,01}{2!}+\ldots+\frac{(0,1)^n}{n!},$$

masala shartiga ko`ra xatolik 0,001 dan katta bo`lmasligi kerak, demak

$R_n(x)=\frac{0,1^{n+1}}{(n+1)!}e^{0,1\theta}<0,001$ tengsizlik o`rinli bo`ladigan birinchi n ni topish yetarli. $e^{0,1\theta}<2$ ekanligini e`tiborga olsak, so`ngi tengsizlikni quyidagicha yozib olish mumkin:

$$\frac{2}{10^{n+1}(n+1)!}<0,001.$$

Endi n=1, 2, 3, ... qiymatlarni so`ngi tengsizlikka qo`yib tekshiramiz va bu tengsizlik n=3 dan boshlab bajarilishini topamiz. Shunday qilib, 0,001 aniqlikda

$$\mathring{a}^{0,1}\approx 1+\frac{0,1}{1!}+\frac{0,01}{2!}+\frac{0,001}{3!}=1,055.$$

Xususiy holda, n=1 bo`lganda

$f(x)\approx f(x_0)+f`(x_0)(x-x_0)$ taqribiy hisoblash formulasi $R_2(x)=\frac{f''(\xi)}{2!}\cdot(x-x_0)^2$, $x_0<\xi<x$ aniqlikda o`rinli bo`ladi.

Misol. Differensial yordamida radiusi r=1,01 bo`lgan doira yuzini toping. Hisoblash xatoligini baholang.

Yechish. Doira yuzi $S=\pi r^2$ ga teng. Bunda $r_0=1$, $\Delta r=0,01$ deb olamiz va S=S(r) funksiya orttirmasini uning differensiali bilan almashtiramiz:

$S(r)\approx S(r_0)+dS(r_0)=S(r_0)+S`(r_0)\Delta r.$

Natijada

$S(1,01)\approx S(1)+dS(1)=S(1)+S`(1)0,01=\pi\cdot 1^2+2\pi\cdot 0,01=1,02\pi$ hosil bo`ladi.

Bunda hisoblash xatoligi

$R_2(r)=\frac{S''(\xi)}{2!}\cdot(r-r_0)^2$, $r_0<\xi<r$ dan katta emas. $S``(r)=2\pi$ va r ga bog`liq emas, shu sababli

$R_2(r)=\frac{2\pi}{2!}\cdot 0,01^2=0,0001\pi$. Demak, hisoblash xatoligi 0,000314 dan katta emas.

Misol. Ushbu $f(x)=e^{x^2-x}$ funksiyaning x=0,03 nuqtadagi qiymatini differensial yordamida hisoblang. Xatolikni baholang.

Yechish. Taqribiy hisoblash formulasi $f(x)\approx f(x_0)+f`(x_0)(x-x_0)$ da $x_0=0$, $x=0{,}03$ qiymatlarni qo`ysak, $f(0{,}03)\approx f(0)+f`(0)0{,}03$ bo`lib, xatolik

$R_2=\frac{f''(\xi)}{2!}\cdot x^2=\frac{f''(\xi)}{2!}\cdot 0{,}03^2$, $0<\xi<0{,}03$ bo`ladi.

Berilgan funksiya hosilalarini va nuqtadagi qiymatlarini hisoblamiz: $f`(x)=(2x-1)\, e^{x^2-x}$, bundan $f`(0)=-1$, $f``(x)=2e^{x^2-x}+(2x-1)^2e^{x^2-x}=e^{x^2-x}(4x^2-4x+3)$, bundan $f``(\xi)<3$. Olingan natijalardan foydalanib, $f(0{,}03)\approx 1+(-1)\cdot 0{,}03=0{,}97$ va $R_2<\frac{3}{2!}\cdot 0{,}03^2=0{,}0017$ ekanligini topamiz.

Teylor formulasi funksiyalarni ekstremumga tekshirishda, qatorlar nazariyasida, integrallarni hisoblashlarda ham keng tatbiqqa ega.

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