Ψ- RIMAN-LIUVILL KASR INTEGRALLARI UCHUN HARDI-LITTLEVUD TENGSIZLIGI HAQIDAGI TEOREMALAR.

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Annotasiya: Lebeg fazosida ψ **-**Riman-Liuvill tipidagi kasrli integralning chegaralanganligi va Hardi-Littlevud tengsizligi haqidagi teoremalari isbotlangan

Kalit so'zlar: Hardi va Littlevud tengsizligi, Riman-Liuvill integrali, *ψ*-Riman-Liuvill kasr integrali, Gyolder tengsizligi, musbat o'suvchi funksiya.

1928-yilda Hardi va Littlevud Riman-Liuvill integralining chegaralanganligini isbotlashdi. Klassik Hardi tengsizligi qism integrali uchun quyidagicha:

$$\left\| x^{\beta-\alpha} \int_0^x \frac{f(y)}{y^{\beta}(x-y)^{1-\alpha}} dy \right\|_{L^p(0,b)} \le C \|f\|_{L^p(0,b)},$$

bunda $0 < \alpha < 1$, $\alpha - \frac{1}{p} < \beta < \frac{1}{q}$, $\frac{1}{q} + \frac{1}{p} = 1$ va $0 < b \le \infty$.

Ushbu mavzuda *ψ*-Riman-Liuvill kasr integrallarining chegaralanganligini ko'rib chiqamiz.

1-teorema. Faraz qilaylik $\alpha > 0, 1 \le p \le \infty, -\infty < a < b < \infty$ va $\psi(x)$ musbat o'suvchi funksiya bo'lsin. U holda $L^p(a,b)$ fazoda $I_{a+}^{\alpha,\psi}$ operator chegaralangan va ushbu

$$\left\| I_{a+}^{\alpha;\psi} \varphi \right\|_{L^p(a,b)} \le \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \|\varphi\|_{L^p(a,b)} \qquad \textbf{(1)}$$

tengsizlik o'rinli bo'ladi.

Isbot. Ataylik $p = \infty$ bo'lsin, u holda ushbu

$$\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} dt \le \frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}$$

tengsizlikka asosan, barcha $x \in (a,b]$ uchun quyidagini olamiz:

$$\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} |\varphi(t)| dt \le \|\varphi\|_{L^\infty(a,b)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} dt \le$$

$$\le \frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha} \|\varphi\|_{L^\infty(a,b)}.$$

p = 1 bo‘lsin, bundan esa quyidagiga ega bo‘lamiz

$$\int_a^b \left| I_{a+}^{\alpha;\Psi}\varphi(x)\right| dx = \int_a^b \left| \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt\right| dx \le$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dtdx$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^b\int_a^x \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dtdx$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^b |u(t)|\int_t^b \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dxdt$$

$$= \frac{1}{\Gamma(\alpha+1)}\int_a^b (\psi(b)-\psi(t))^{\alpha}|\varphi(t)|\,dt \le \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\int_a^b |\varphi(t)|dt$$

Endi $1<p<\infty$ bo‘lgan holatni ko‘rib chiqamiz. $q>0$ va $\frac{1}{p}+\frac{1}{q}=1$ bo‘lsin.

Gyolder tengsizligiga asosan, quyidagiga ega bo‘lamiz

$$\left| I_{a+}^{\alpha;\psi}\varphi(x)\right| = \left| \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt\right|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{1}{q}}[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{1}{p}}|\varphi(t)|\,dt \le$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}dt\right)^{1/q}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p\,dt\right)^{1/p} \le$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{1/q}\left(\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|u(t)|^p\,dt\right)^{1/p}.$$

Shunday qilib, $\psi'(x)$ o‘suvchi ekanligini hisobga olsak, quyidagini olamiz

$$\int_a^b \left| I_{a+}^{\alpha;\Psi}\varphi(x)\right|^p dx \le \frac{1}{[\Gamma(\alpha)]^p}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{p/q}\int_a^b\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p\,dtdx \le$$

$$\le \frac{1}{[\Gamma(\alpha)]^p}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{p/q}\int_a^b\int_a^x \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|^p\,dtdx$$

$$= \frac{1}{[\Gamma(\alpha)]^p}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{p/q}\int_a^b |\varphi(t)|^p\int_t^b \psi'(x)(\psi(x)-\psi(t))^{\alpha-1}dxdt$$

$$\le \frac{1}{[\Gamma(\alpha)]^p}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{p/q}\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\int_a^b |\varphi(t)|^p\,dt$$

$$= \frac{1}{[\Gamma(\alpha)]^p}\left(\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\right)^{\frac{p}{q}+1}\frac{(\psi(b)-\psi(a))^{\alpha}}{\alpha}\int_a^b |\varphi(t)|^p\,dt.$$

Demak, oxirgi tengsizlikdan (1) kelib chiqadi. □

2-teorema. Faraz qilaylik $0<\alpha<1, -\infty<a<b<\infty$ va $\psi(x)$ musbat o‘suvchi funksiya bo‘lsin. Agar $1\le p<\frac{1}{\alpha}$ va $1\le q<\frac{p}{1-\alpha p}$, $\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}$ bo‘lsa, u holda $I_{a+}^{\alpha,\psi}$ operator $L^p(a,b)$ fazoni $L^q(a,b)$ fazoga ko‘chiradi va ushbu

$$\left\| \mathrm{I}_{a+}^{\alpha;\psi} u \right\|_{L^q(\alpha,b)} \le \frac{1}{\Gamma(\alpha)} \left(\frac{\left[\psi'(b)\right]^{r-1} \left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r} \right)^{1-\frac{1}{p}+\frac{1}{q}} \|u\|_{L^p(\alpha,b)}$$

tengsizlik o‘rinli bo‘ladi.

Isboti. Endi p = 1 va $1\le q<\frac{1}{1-\alpha}$ bo‘lgan holatini ko‘rib chiqamiz. q = 1 bo‘lgan hol 1-teoremada isbotlangan, shuning uchun $1\le q<\frac{1}{1-\alpha}$ holni qaraymiz. Gyolder tengsizligiga asosan, quyidagiga ega bo‘lamiz

$$\left| I_{a+}^{\alpha;\psi} \varphi(x) \right| \le \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} |\varphi(t)| dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} |\varphi(t)|^{\frac{1}{q}} |\varphi(t)|^{1-\frac{1}{q}} dt$$

$$\le \frac{1}{\Gamma(\alpha)} (\int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^q |\varphi(t)| dt)^{1/q} \left(\int_a^x |\varphi(t)| dt \right)^{1-\frac{1}{q}}.$$

Shundan so‘ng,

$$\left| \mathrm{I}_{a+}^{\alpha,\psi} \varphi(x) \right|^q \le \frac{1}{[\Gamma(\alpha)]^q} \int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha}-1]^q |\varphi(t)| dt \|\varphi\|_{L^1(a,b)}^{q-1}$$

Bundan esa,

$$\left\| \mathrm{I}_{a+}^{\alpha,\psi} \varphi(x) \right\|_{L^p(a,b)}^q \le \frac{1}{[\Gamma(\alpha)]^q} \int_a^b \int_a^x [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}-1]^q |\varphi(t)| dtdx \|\varphi\|_{L^1(a,b)}^{q-1}$$

$$\le \frac{1}{[\Gamma(\alpha)]^q} \int_a^b |\varphi(t)| \int_t^b [\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^q dxdt \|\varphi\|_{L^1(a,b)}^{q-1}$$

$$= \frac{1}{[\Gamma(\alpha)]^q} \int_a^b |\varphi(t)| [\psi'(b)]^{q-1} \int_t^b \psi'(x)(\psi(x)-\psi(t))^{(\alpha-1)q} dxdt \|\varphi\|_{L^1(a,b)}^{q-1}$$

$$= \frac{[\psi'(b)]^{q-1}}{[\Gamma(\alpha)]^q} \int_a^b |u(t)| \frac{(\psi(b)-\psi(t))^{1-(1-\alpha)q}}{1-(1-\alpha)q} dt \|\varphi\|_{L^1(a,b)}^{q-1}$$

$$\le \frac{[\psi'(b)]^{q-1}(\psi(b)-\psi(a))^{1-(1-\alpha)q}}{(1-(1-\alpha)q)[\Gamma(\alpha)]^q} \|\varphi\|_{L^1(a,b)}^q.$$

Demak, $\forall 1\le q<\frac{1}{1-\alpha}$ uchun

$$\left|I_{a+}^{\alpha;\Psi}u\right|_{L^q_{(a,b)}} \le \left(\frac{[\psi'(b)]^{q-1}(\psi(b)-\psi(a))^{1-(1-\alpha)q}}{1-(1-\alpha)q[\Gamma(\alpha)]^q}\right)^{1/q}\|\varphi\|_{L^1_{(a,b)}}$$ bo‘ladi.

Isbotni davom ettirib, $1<p<\frac{1}{\alpha}$ va $p\le q<\frac{p}{1-\alpha p}$ holni ko‘rib chiqamiz. $p=q$ hol 1-teoremada isbotlangan, shuning uchun biz faqat ishda $p<q<\frac{p}{1-\alpha p}$ holni qaraymiz. Quyidagi belgilashni kiritamiz $\delta=\frac{1}{p}-\frac{1}{q}$,

bunda $0<\delta<\alpha<1$. r ni topamiz

$$\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}=1-\delta\Rightarrow r=\frac{1}{1-\delta}.$$

$r<q$ e’tiborga olsak, Gyolder tengsizligini qo‘llaymiz

$$\left|I_{a+}^{\alpha;\Psi}\varphi(x)\right|=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|\varphi(t)\right|dt=$$

$$=\frac{1}{\Gamma(\alpha)}\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{\frac{r}{q}}\left|\varphi(t)\right|^{\frac{p}{q}}[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^{1-\frac{r}{q}}\left|\varphi(t)\right|^{p(\frac{1}{p}-\frac{1}{q})}dt$$

$$\le\frac{1}{\Gamma(\alpha)}\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r\left|\varphi(t)\right|^p dt\right)^{1/q}\left(\int_a^x\left|\varphi(t)\right|^p dt\right)^{\frac{1}{p}-\frac{1}{q}}$$

$$\times\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r dt\right)^{1-\frac{1}{p}}=\frac{1}{\Gamma(\alpha)}\left(\frac{[\psi'(x)]^{r-1}(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}}$$

$$\times\left(\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r\left|\varphi(t)\right|^p dt\right)^{1/q}\left(\int_a^x\left|\varphi(t)\right|^p dt\right)^{\frac{1}{p}-\frac{1}{q}}$$

Teorema shartiga asosan, oxirgi ifoda quyidagi shartlarda o‘rinli bo‘ladi:

$$\frac{1}{r}=1-\delta>1-\alpha\Rightarrow(1-\alpha)r<1.$$

Bu shartga ko‘ra, quyidagi tengsizlik o‘rinli bo‘ladi

$$\left\|I_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^q(a,b)}^q=\int_a^b\left|I_{\alpha+}^{\alpha;\psi}\varphi(x)\right|^q dx$$

$$\le\frac{1}{[\Gamma(\alpha)]^q}\int_a^b\left[\int_a^x[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}]^r\left|\varphi(t)\right|^p dt\,\|\varphi\|_{L^p(a,b)}^{q-p}\right.$$

$$\left.\left(\frac{[\psi'(x)]^{r-1}(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\right]dx$$

$$\leq \frac{\|\varphi\|^{q-p}_{L^p(a,b)}}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\int_a^b\int_a^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^r|\varphi(t)|^p\,dtdx$$

$$\leq \frac{\|\varphi\|^{q-p}_{L^p(a,b)}}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})}\frac{[\psi'(b]^{r-1}(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\|\varphi\|^{q-p}_{L^p(a,b)}$$

$$=\frac{1}{[\Gamma(\alpha)]^q}\left(\frac{[\psi'(b)]^{r-1}C(\psi(b)-\psi(\alpha))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q-(1-\frac{1}{p})+1}\|\varphi\|^q_{L^p(a,b)}$$

Demak, $\forall p\leq q<\dfrac{p}{1-\alpha p}$ uchun

$$\left\|I^{\alpha;\psi}_{\alpha+}\varphi\right\|_{L^q(\alpha,b)}\leq\frac{1}{\Gamma(\alpha)}\left(\frac{[\psi'(b)]^{r-1}(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\|\varphi\|_{L^p(a,b)}$$ bo‘ladi.

3-teorema. Faraz qilaylik $0<\alpha<1, -\infty<a<b<\infty$ va $\psi(x)$ musbat o‘suvchi funksiya bo‘lsin. Agar $\alpha=\dfrac{1}{p}$, $1\leq q<\infty)$ va $\dfrac{1}{r}=1+\dfrac{1}{q}-\dfrac{1}{p}$ bo‘lsa, u holda $I^{\alpha,\psi}_{a+}$ operator $L^p(a,b)$ fazoni $L^q(a,b)$ fazoga ko‘chiradi va ushbu

$$\left\|I^{\alpha;\psi}_{\alpha+}\varphi\right\|^q_{L^q(a,b)}\leq\frac{1}{\Gamma(\alpha)}\left([\psi'(b)]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\|\varphi\|_{L^p(a,b)}$$

tengsizlik o‘rinli bo‘ladi.

Isboti. $p=\dfrac{1}{\alpha}$ va $\dfrac{1}{\alpha}\leq q<\infty$ bo‘lgan holatni qaraymiz. Faqat $\dfrac{1}{\alpha}<q<\infty$ bo‘lgan holda qaraymiz, chunki (1) ga ko‘ra, $p=q$ hol esa isbotlangan. δ va r larni quyidagicha aniqlaymiz

$$\delta=\frac{1}{p}-\frac{1}{q}=\alpha-\frac{1}{q}\qquad\text{va}\qquad\frac{1}{r}=1-\delta=1+\frac{1}{q}-\frac{1}{p}.$$

bunda $0<\delta<\alpha<1$ va $r<\dfrac{1}{1-\alpha}$ bajarilsin. Gyolder tengsizligiga ko‘ra, quyidagiga ega bo‘lamiz

$$\left|I^{\alpha;\psi}_{\alpha+}\varphi(x)\right|\leq\frac{1}{\Gamma(\alpha)}\int_\alpha^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}|\varphi(t)|\,dt$$

$$=\frac{1}{\Gamma(\alpha)}\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{\frac{r}{q}}|\varphi(t)|^{\frac{p}{q}}\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r(1-\frac{p}{q})}|\varphi(t)|^{1-\frac{p}{q}}dt$$

$$\leq\frac{1}{\Gamma(\alpha)}\left(\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^r|\varphi(t)|^p dt\right)^{1/q}\times$$

$$\left(\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r(1-\frac{1}{p})\theta}|\varphi(t)|^{r(1-\frac{p}{q})\theta}dt\right)^{1/\theta},\frac{1}{q}+\frac{1}{\theta}=1. \qquad \textbf{(2)}$$

Boshqa tomondan Gyolder tengsizligidan quyidagiga kelamiz,

$$\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r(1-\frac{1}{p})\theta}|\varphi(t)|^{r(1-\frac{p}{q})\theta}dt\leq$$

$$\leq\left(\int_a^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r(1-\frac{1}{p})\theta s}dt\right)^{1/s}\left(\int_a^x|\varphi(t)|^{\left(1-\frac{p}{q}\right)\theta m}dt\right)^{1/m},\frac{1}{s}+\frac{1}{m}=1,$$

Quyidagi belgilashlarni olamiz

$m=\dfrac{p(q-1)}{q-p},\theta=\dfrac{q}{q-1}$ va $s=\dfrac{p(q-1)}{q(p-1)}\Rightarrow r\left(1-\dfrac{1}{p}\right)\theta s=r$ va $\left(1-\dfrac{1}{p}\right)\theta m=p$.

Shuning uchun, quyidagiga ega bo‘lamiz

$$\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r(1-\frac{1}{p})\theta}|\varphi(t)|^{r(1-\frac{p}{q})\theta}dt$$

$$\leq\left(\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]\right)^{\frac{q(p-1)}{p(q-1)}}\left(\int_a^x|\varphi(t)|^p dt\right)^{\frac{q-p}{p(q-1)}} \qquad \textbf{(3)}$$

$$=\left([\psi'(x)]^{r-1}\frac{(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{\frac{q(p-1)}{p(q-1)}}\left(\int_a^x|\varphi(t)|^p dt\right)^{\frac{q(p-1)}{p(q-1)}}$$

Endi **(2)** ni **(3)** ni o‘rniga qo‘yish orqali, quyidagini olamiz

$$\left|I_{\alpha+}^{\alpha;\psi}\varphi(x)\right|\leq\frac{1}{\Gamma\leq(\alpha)}\left(\int_\alpha^x\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^r|\varphi(t)|^p dt\right)^{1/q}\left([\psi'(x)]^{r-1}\frac{(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{\frac{p-1}{p}}$$

$$\times\left(\int_a^x|\varphi(t)^p dt|\right)^{\frac{q-p}{pq}}.$$

Shuningdek

$$\left|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi(x)\right| \le \frac{1}{\Gamma \le (\alpha)^{q}}\left(\left[\psi'(x)\right]^{r-1}\frac{\left(\psi(x)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q\left(1-\frac{1}{p}\right)}\int_{\alpha}^{x}\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r}\left|\varphi(t)\right|^{p}dt$$

$$\times\left\|\varphi\right\|_{L^{p}(a,b}^{q-p}.$$

Shuning uchun, quyidagiga ega bo‘lamiz

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^{q}(a,b)}^{q}=\int_{\alpha}^{b}\left|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi(x)\right|^{q}dx$$

$$\le\frac{1}{\left[\Gamma(\alpha)\right]^{q}}\left\|\varphi\right\|_{L^{p}(a,p)}^{q-p}\left(\left[\psi'(x)\right]^{r-1}\frac{\left(\psi(x)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q\left(1-\frac{1}{p}\right)}\int_{\alpha}^{x}\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r}\left|\varphi(t)\right|^{p}dtdx$$

$$\le\frac{\left\|\varphi\right\|_{L^{p}(a,p)}^{q-p}}{\left[\Gamma(\alpha)\right]^{q}}\left(\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q\left(1-\frac{1}{p}\right)}\int_{a}^{b}\int_{\alpha}^{x}\left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^{r}\left|\varphi(t)\right|^{p}dtdx$$

$$\le\frac{\left\|\varphi\right\|_{L^{p}(a,p)}^{q-p}}{\left[\Gamma(\alpha)\right]^{q}}\left(\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{q\left(1-\frac{1}{p}\right)}\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\left\|\varphi\right\|_{L^{p}(a,p)}^{q-p}$$

$$=\frac{\left\|\varphi\right\|_{L^{p}(a,p)}^{q}}{\left[\Gamma(\alpha)\right]^{q}}\left(\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1+q\left(1-\frac{1}{q}\right)}$$

Bunda, $\forall p=\frac{1}{\alpha}\le q<\infty$

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^{q}(a,b)}^{q}\le\frac{1}{\Gamma(\alpha)}\left(\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\left\|\varphi\right\|_{L^{p}(a,b)}.$$

3-teorema. Faraz qilaylik $0<\alpha<1, -\infty<a<b<\infty$ va $\psi(x)$ musbat o‘suvchi funksiya bo‘lsin. Agar $\alpha\in[\frac{1}{p},1)$, $p\le q\le\infty$ bo‘lsa, u holda $I_{a+}^{\alpha,\psi}$ operator $L^{p}(a,b)$ fazoni $L^{q}(a,b)$ fazoga ko‘chiradi va ushbu

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^{q}(a,b)}^{q}\le\frac{1}{\Gamma(\alpha)}\left(\left[\psi'(b)\right]^{r-1}\frac{\left(\psi(b)-\psi(a)\right)^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}}\left\|\varphi\right\|_{L^{p}(a,b)}$$

tengsizlik o‘rinli bo‘ladi.

Isbot. Endi biz $\frac{1}{\alpha}<p<\infty$ va $p\le q\le\infty$ bo‘lgan holatni ko‘rib chiqamiz

$q = p$ ni **(1)** da ko‘rib chiqildi, shuning uchun biz faqat $p < q \le \infty$ da ko‘rib chiqamiz. Biz $p < q < \infty$ ni ko‘rib chiqishdan boshlaymiz. Ushbu shartlar

$$\delta = \frac{1}{p} - \frac{1}{q} \quad \text{va} \quad \frac{1}{r} = 1 - \delta = 1 + \frac{1}{q} - \frac{1}{p}$$

bajarilsin. Shuningdek

$0 < \delta < \alpha$ va $1 < r < \dfrac{1}{1-\alpha}$.

Keyin, $$\left|I_{\alpha+}^{\alpha;\psi} u(x)\right| \le \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(t) - \psi(t))^{\alpha-1} |u(t)|\, dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{\frac{r}{q}} |u(t)|^{\frac{p}{q}} \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{1-\frac{r}{q}} |u(t)|^{p\left(\frac{1}{p}-\frac{1}{q}\right)} dt$$

Chunki $1 - \dfrac{r}{q} = r\left(1 - \dfrac{1}{p}\right)$, Gyolder tengsizligidan kelib chiqadi

$$\left|I_{\alpha+}^{\alpha;\psi} \varphi(x)\right| \le \frac{1}{\Gamma(\alpha)} \int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{\frac{r}{q}} |\varphi(t)|^{\frac{p}{q}} \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r\left(1-\frac{1}{p}\right)} |\varphi(t)|^{p\left(\frac{1}{p}-\frac{1}{q}\right)} dt$$

$$\le \frac{1}{\Gamma(\alpha)} \left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r} |\varphi(t)|^{p}\, dt\right)^{1/q}$$

$$\left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r\theta\left(1-\frac{1}{p}\right)} |\varphi(t)|^{p\theta\left(\frac{1}{p}-\frac{1}{q}\right)} dt\right)^{\frac{1}{\theta}}, \tag{4}$$

bunda $\dfrac{1}{q} + \dfrac{1}{\theta} = 1$. Boshqa tarafdan

$$\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r\theta\left(1-\frac{1}{p}\right)} |\varphi(t)|^{p\theta\left(\frac{q-p}{pq}dt\right)} dt \le \left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r\theta m\left(1-\frac{1}{p}\right)} dt\right)^{\frac{1}{m}} \ddot{x}$$

$$\times \left(\int_a^x |\varphi(t)|^{p\theta n\left(\frac{1}{p}-\frac{1}{q}\right)} dt\right)^{1/n}, \frac{1}{m} + \frac{1}{n} = 1 \tag{5}$$

Shuni yodda tutish kerak, $\theta = \dfrac{q}{q-1}$, $m = \dfrac{p(q-1)}{q(p-1)}$ va $n = \dfrac{p(q-1)}{q-p}$.

Shunday qilib **(5)** ni **(4)** ga olib borib qo‘yib quyidagini olamiz.

$$\left|I_{\alpha+}^{\alpha;\psi} \varphi(x)\right| \le \frac{1}{\Gamma(\alpha)} \left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r} |\varphi(t)|^{p}\, dt\right)^{1/q} \left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^{r}\right)^{\frac{p-1}{p}}$$

$$\times\left(\int_a^x |\varphi(t)|^p\,dt\right)^{\frac{q-p}{pq}}$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^r |\varphi(t)|^p\,dt\right)^{1/q}\left(\frac{\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^r}{1-(1-\alpha)r}\right)^{\frac{p-1}{p}} \times\|\varphi\|_{L^p(a,b)}^{\frac{q-p}{q}}$$

Shuningdek,

$$\left|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi(x)\right|^q \le$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^r |\varphi(t)|^p\,dt\left(\frac{\int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^r}{1-(1-\alpha)r}\right)^{\frac{q(p-1)}{p}} \|\varphi\|_{L^p(a,b)}^{q-p}$$

So‘ngra, quyidagiga ega bo‘lamiz

$$\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^p(a,b)}^q = \int_a^b \left|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi(x)\right|^q dx$$

$$\le \frac{\|\varphi\|_{L^p(a,b)}^{q-p}}{\left[\Gamma(\alpha)\right]^q}\int_a^b\left(\left[\psi'(x)\right]^{r-1}\frac{(\psi(x)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{\frac{q(p-1)}{p}} \int_a^x \left[\psi'(t)(\psi(t)-\psi(t))^{\alpha-1}\right]^r |\varphi(t)|^p\,dtdx$$

$$\le \frac{1}{\left[\Gamma(\alpha)\right]^q}\left(\left[\psi'(b)\right]^{r-1}\frac{(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1+\frac{q(p-1)}{p}} \|\varphi\|_{L^p(a,b)}^q$$

Bunda, $\left\|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi\right\|_{L^p(a,b)}^q \le \frac{1}{\Gamma(\alpha)}\left(\left[\psi'(b)\right]^{r-1}\frac{(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1-\frac{1}{p}+\frac{1}{q}} \|\varphi\|_{L^p(a,b)}.$

Endi biz $\frac{1}{\alpha}<p<\infty$ va $q=\infty$ bo‘lgan holni qaraymiz. Gyolder tengsizligidan foydalanib quyidagini olamiz:

$$\left|\mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi(x)\right| \le \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(t)-\psi(t))^{\alpha-1}|\varphi(t)|^p\,dt$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\int_a^x \left[\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\right]^r dt\right)^{1/r}\left(\int_a^x |\varphi(t)|^p\,dt\right)^{1/p}$$

$$\le \frac{1}{\Gamma(\alpha)}\left(\left[\psi'(b)\right]^{r-1}\frac{(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r}\right)^{1/r}\|\varphi\|_{L^p(a,b)}.$$

Bu tengsizlikdan quyidagiga ega bo‘lamiz

$$\left\| \mathrm{I}_{\alpha+}^{\alpha;\psi}\varphi \right\|_{L^{\infty}(a,b)} \le \frac{1}{\Gamma(\alpha)} \left(\left[\psi'(b) \right]^{r-1} \frac{(\psi(b)-\psi(a))^{1-(1-\alpha)r}}{1-(1-\alpha)r} \right)^{1-\frac{1}{p}} \left\| \varphi \right\|_{L^{p}(a,b)} \quad . \square$$

ADABIYOTLAR:

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