ON THE NUMBER AND LOCATION OF EIGENVALUES OF THE DISCRETE SCHRÖDINGER OPERATOR ON A LATTICE

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Abstract. *We study spectral properties of the discrete Schrödinger operators $H_{\lambda\mu}$ associated to a system of two identical particles with an indefinite sign interaction potential with energy $\lambda \in \mathbb{R}$ on nearest neighboring sites and energy $\mu \in \mathbb{R}$ on the next nearest neighboring sites on the one-dimensional lattice $\mathbb{Z}$. We obtain the exact conditions on the parameters for the operator to have zero, one or two eigenvalues outside the essential spectrum.*

Key words and phrases: eigenvalues, essential spectrum, discrete spectrum.

Let $\ell^{2,o}(\mathbb{Z})$ be the Hilbert space of square summable odd functions on $\mathbb{Z}$. The one-particle discrete Schrödinger operator $\widehat{H}_{\lambda\mu}$, which are dependent on real parameters λ and μ, acting in $\ell^{2,o}(\mathbb{Z})$ is given as

$$\widehat{H}_{\lambda\mu} = \widehat{H}_0 + \widehat{V}_{\lambda\mu},$$

where $\widehat{H}_0$ is the convolution-type operator

$$(\widehat{H}_0\hat{\varphi})(x) = \sum_{s\in Z} \hat{\varepsilon}(s)\hat{\varphi}(x+s), \qquad \hat{\varphi} \in \ell^{2,o}(\mathbb{Z})$$

and $\widehat{V}_{\lambda\mu}$ is a multiplication operator

$$(\widehat{V}_{\lambda\mu}\hat{\varphi})(x) := \hat{v}_{\lambda\mu}(x)\hat{\varphi}(x), \quad \hat{\varphi} \in \ell^{2,o}(\mathbb{Z}),$$

where, the functions $\hat{\varepsilon}(s)$ and $\hat{v}_{\lambda\mu}(s)$ are defined on $\mathbb{Z}$ as

$$\hat{\varepsilon}(s) = \begin{cases} 1, & |s| = 0 \\ -\frac{1}{2}, & |s| = 1 \\ 0, & |s| > 1 \end{cases}$$

and

$$\hat{v}_{\lambda\mu}(s) = \begin{cases} \frac{\mu}{2}, & |s| = 1 \\ \frac{\lambda}{2}, & |s| = 2 \\ 0, & |s| \neq 1,2. \end{cases}$$

We remark that $\widehat{H}_{\lambda\mu}$ is a bounded self-adjoint operator on $\ell^{2,o}(\mathbb{Z})$.

Let $L^{2,o}(\mathbb{T}, d\nu)$ be the Hilbert space of square integrable odd functions on $\mathbb{T}$, with $d\nu$ being a (normalized) Haar measure on $\mathbb{T}$, $\left(d\nu(p) = \frac{dp}{2\pi}\right)$.

In the momentum representation, the operator $H_{\lambda\mu}$ acts on $L^{2,o}(\mathbb{T}, d\nu)$ as

$$H_{\lambda\mu} = H_0 + V_{\lambda\mu},$$

where H_0 is the multiplication operator by function $\varepsilon(p) = 1 - \cos p$

$$(H_0 f)(p) = \varepsilon(p) f(p), \quad f \in L^{2,o}(\mathbb{T}, d\nu),$$

and $V_{\lambda\mu}$ is the integral operator of rank 2

$$(V_{\lambda\mu} f)(p) = \int_{\mathbb{T}} (\mu \sin p \sin t + \lambda \sin 2p \sin 2t) \, f(t) \, d\nu(t) \ , \quad f \in L^{2,o}(\mathbb{T}, d\nu).$$

The perturbation operator $V_{\lambda\mu}, \lambda, \mu \in \mathbb{R}$ is of rank 2, therefore by the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\lambda\mu})$ of $H_{\lambda\mu}$ does not depend on $\lambda, \mu \in \mathbb{R}$ and coincides with the spectrum $\sigma(H_0)$ of H_0, i.e.,

$$\sigma_{ess}(H_{\lambda\mu}) = \sigma(H_0) = [\min_{p\in\mathbb{T}} \varepsilon(p), \max_{p\in\mathbb{T}} \varepsilon(p)] = [0,2].$$

To formulate the main theorem, we introduce the following regions $\mathbb{G}_{2,+}, \mathbb{G}_{1,+}$ and $\mathbb{G}_{0,+}$ associated to the function $C_0^+(\lambda, \mu)$

$$\mathbb{G}_{2,+} = \{(\lambda, \mu) \in \mathbb{R}^2 : C_0^+(\lambda, \mu) > 0, \quad \mu > 2\},$$
$$\mathbb{G}_{1,+} = \{(\lambda, \mu) \in \mathbb{R}^2 : C_0^+(\lambda, \mu) < 0\},$$
$$\mathbb{G}_{0,+} = \{(\lambda, \mu) \in \mathbb{R}^2 \ C_0^+(\lambda, \mu) > 0, \quad \mu < 2\}$$

and the regions $\mathbb{G}_{2,-}, \mathbb{G}_{1,-}$ and $\mathbb{G}_{0,-}$ associated to the function $C_0^-(\lambda, \mu)$

$$\mathbb{G}_{2,-} = \{(\lambda, \mu) \in \mathbb{R}^2 : C_0^-(\lambda, \mu) > 0, \quad \mu < -2\},$$
$$\mathbb{G}_{1,-} = \{(\lambda, \mu) \in \mathbb{R}^2 : C_0^-(\lambda, \mu) < 0\},$$
$$\mathbb{G}_{0,-} = \{(\lambda, \mu) \in \mathbb{R}^2 : C_0^-(\lambda, \mu) > 0, \quad \mu > -2\}.$$

Theorem 1.

(i) For $(\lambda, \mu) \in G_{02} = \mathbb{G}_{0,-} \cap \mathbb{G}_{2,+}$, the operator $H_{\lambda\mu}$ has no eigenvalues below the essential spectrum and has two eigenvalues $\zeta_1(\lambda, \mu)$ and $\zeta_2(\lambda, \mu)$ satisfying the relations

$$2 < \zeta_1(\lambda, \mu) < \zeta_{\min}(\lambda, \mu) \le \zeta_{\max}(\lambda, \mu) < \zeta_2(\lambda, \mu).$$

(ii) For $(\lambda, \mu) \in G_{01} = \mathbb{G}_{0,-} \cap \mathbb{G}_{1,+}$, the operator $H_{\lambda\mu}$ has no eigenvalues below the essential spectrum and it has an eigenvalue $\zeta_2(\lambda, \mu)$ satisfying the relation

$$\zeta_2(\lambda, \mu) > 2.$$

(iii) For $(\lambda, \mu) \in G_{00} = \mathbb{G}_{0,-} \cap \mathbb{G}_{0,+}$, the operator $H_{\lambda\mu}$ has no eigenvalues below and above the essential spectrum.

(iv) For $(\lambda, \mu) \in G_{11} = \mathbb{G}_{1,-} \cap \mathbb{G}_{1,+}$, the operator $H_{\lambda\mu}$ has two eigenvalues $\zeta_1(\lambda, \mu)$ and $\zeta_2(\lambda, \mu)$ satisfying the following relations

$$\zeta_1(\lambda, \mu) < 0 \quad and \quad \zeta_2(\lambda, \mu) > 2.$$

(v) For $(\lambda, \mu) \in G_{10} = \mathbb{G}_{1,-} \cap \mathbb{G}_{0,+}$, the operator $H_{\lambda\mu}$ has one negative eigenvalue $\zeta_1(\lambda, \mu)$ and has no eigenvalues above the essential spectrum.

(vi) For $(\lambda, \mu) \in G_{20} = \mathbb{G}_{2,-} \cap \mathbb{G}_{0,+}$, the operator $H_{\lambda\mu}$ has two eigenvalues $\zeta_1(\lambda, \mu)$ and $\zeta_2(\lambda, \mu)$ satisfying the relations

$$\zeta_1(\lambda, \mu) < \zeta_{\min}(\lambda, \mu) \le \zeta_{\max}(\lambda, \mu) < \zeta_2(\lambda, \mu) < 0$$

and has no eigenvalues above the essential spectrum.

Theorem 2.

*(i)*If $(\lambda_1,\mu_1) \in \{(\lambda,\mu) \in \mathbb{R}^2: C_0^+(\lambda,\mu) = 0,\ \lambda > 1,\ \mu > 2\ or\ C_0^-(\lambda,\mu) = 0,\ \lambda > |\frac{1}{\sqrt{2}}|,\ \mu > -2,\ \}$, then the operator $H_{\lambda_1\mu_1}$ has no eigenvalues below and it has one eigenvalue above the essential spectrum.

*(ii)*If $(\lambda_1,\mu_1) \in \{(\lambda,\mu) \in \mathbb{R}^2: C_0^+(\lambda,\mu) = 0,\ \lambda \le |\frac{1}{\sqrt{2}}|,\ \mu < 2\ or\ C_0^-(\lambda,\mu) = 0,\ \lambda \le |\frac{1}{\sqrt{2}}|,\ \mu > -2,\ \}$, then the operator $H_{\lambda_1\mu_1}$ has no eigenvalues outside the essential spectrum.

*(iii)*If $\lambda_1,\mu_1) \in \{(\lambda,\mu) \in \mathbb{R}^2: C_0^+(\lambda,\mu) = 0,\ \lambda > |\frac{1}{\sqrt{2}}|,\ \mu < 2\ or\ C_0^-(\lambda,\mu) = 0,\ \lambda < -1,\ \mu < -2,\ \}$, then the operator $H_{\lambda_1\mu_1}$ has one eigenvalue below and it has no eigenvalues above the essential spectrum.

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