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THE NUMBER OF EIGENVALUES OF THE DISCRETE SCHRÖDINGER OPERATOR

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Abstract: *We study the discrete Schrodinger operators $H_{\gamma\lambda\mu}$ of a quantum particle moving in the one-dimensional lattice Z interacting with an indefinite sign external field, where the potential has both positive and negative values. In this paper we consider the family $H_{\gamma\lambda\mu}$ of Schrodinger operators, associated to the Bose-Hubbard Hamiltonian of a single boson on the one dimensional lattice Z with on-site interaction $\gamma \in R$, nearest-neighbor interaction $\lambda \in R$ and next nearest-neighbor interaction $\mu \in R$. We find suffiently conditions for the parameters , such that the operator $H_{\gamma\lambda\mu}$ has no one eigenvalues below the essential spectrum and three eigenvalues above the essential spectrum.*

Introduction. We study the discrete Schrodinger operators $H_{\gamma\lambda\mu}$ of a quantum particle moving in the one-dimensional lattice Z interacting with an indefinite sign external field, where the potential has both positive and negative values. We also find particular connected components of the $\gamma - \lambda - \mu$ space, where the number of eigenvalues are preserved. Interestingly, this division is only dependent on the first two coefficients of the asymptotic expansion of the Fredholm determinant corresponding to the operator $H_{\gamma\lambda\mu}$ near the threshold (the left edge of the essential spectrum).

Research methods. To study the discrete spectrum of $H_{\gamma\lambda\mu}$we introduce the Fredholm determinant $\Delta_{\gamma\lambda\mu}(z)$ associated to the $H_{\gamma\lambda\mu}$in $l^{2,e}(Z)$. It is well-known that eigenvalues of $H_{\gamma\lambda\mu}$are zeros of this determinant, and vice versa (see Lemma 1). Moreover, the number of zeros of $\Delta_{\gamma\lambda\mu}(z)$ can change if and only if its asymptotics as z approaches to the thresholds of the essential spectrum vanish.

Results and discussions. We partition the (γ, λ, μ) –space of interactions into connected components by means the equation $C^{+}(\gamma, \lambda, \mu) = 0$, where the function $C^{+}(\gamma, \lambda, \mu)$ appear as constant in front of the main term in the asymptotics of the Fredholm determinant $\Delta_{\gamma\lambda\mu}(z)$, as z converges to the edges of the essential spectrum (Lemma 4). As we noticed above, the number of eigenvalues of $H_{\gamma\lambda\mu}$ changes if and only if the ”constant” $C^{+}(\gamma, \lambda, \mu)$ changes its sign . Hence, while the point (γ, λ, μ) runs in R^3 and does not cross any of these surface, no qualitative or quantitative changes occur in the discrete spectrum of $H_{\gamma\lambda\mu}$ however, as soon as (γ, λ, μ) crosses any of those surface the essential spectrum.

Conclusion. Since H_γλμ is of rank at most three, by the min-max principle, it has at most three eigenvalues outside the essential spectrum. In this paper, we show that the operator H_γλμ has three eigenvalues above the essential spectrum, when the parameters satisfy certain conditions.

Keywords: *Schrödinger operator, essential spectrum, one-dimensional lattice*

The coordinate representation.

Let Z be the one-dimensional lattice and $l^{2,e}(Z)$ be the Hilbert space of square summable even functions on Z. The one-particle discrete Schrödinger operator $\widehat{H}_{\gamma\lambda\mu}$, which are dependent on real parameters γ, λ and μ, acting in $l^{2,e}(Z)$ is given as

$$\widehat{H}_{\gamma\lambda\mu} = \widehat{H}_0 + \widehat{V}_{\gamma\lambda\mu},$$

where $\widehat{H}_0$ is the Teoplitz type operator

$$(\widehat{H}_0\hat{\varphi})(x) = \sum_{s\in Z} \hat{\varepsilon}(s)\hat{\varphi}(x+s), \ \hat{\varphi} \in l^{2,e}(Z)$$

and $\widehat{V}_{\gamma\lambda\mu}$ is a multiplication operator

$$(\widehat{V}_{\gamma\lambda\mu}\hat{\varphi})(x) := \hat{v}_{\gamma\lambda\mu}(x)\hat{\varphi}(x), \quad \hat{\varphi} \in l^{2,e}(Z),$$

where, the functions $\hat{\varepsilon}(s)$ and $\hat{v}_{\gamma\lambda\mu}(s)$ are defined on Z as

$$\hat{\varepsilon}(s) = \{1, \quad |s| = 0 \ -\frac{1}{2}, \quad |s| = 1 \ 0, \quad |s| > 1 \quad and \quad \hat{v}_{\gamma\lambda\mu}(s) = \{\gamma, \quad |s| = 0 \ \frac{\lambda}{2}, \quad |s| = 1 \ \frac{\mu}{2}, \quad |s| = 2 \ 0, \quad |s| > 2,$$

respectively. We remark that $\widehat{H}_{\gamma\lambda\mu}$ is a bounded self-adjoint operator on $l^{2,e}(Z)$.

The momentum representation

Let $T = (-\pi;\pi]$ be a one-dimensional torus and $L^{2,e}(T,dv)$ be the Hilbert space of square integrable even functions on T, with dv being a (normalized) Haar measure on T, $\left(dv(p) = \frac{dp}{2\pi}\right)$.

In the momentum representation, the operator $H_{\gamma\lambda\mu}$ acts on $L^{2,e}(T,dv)$ as

$$H_{\gamma\lambda\mu} = H_0 + V_{\gamma\lambda\mu},$$

where H_0 is the multiplication operator by function $\varepsilon(p) = 1 - cosp$

$$(H_0 f)(p) = \varepsilon(p)f(p), \quad f \in L^{2,e}(T,dv),$$

and $V_{\gamma\lambda\mu}$ is the integral operator of rank at most three

$$(V_{\gamma\lambda\mu}f)(p) = \int_T (\gamma + \mu cospcost + \lambda cos2pcos2t)f(t)dv(t), \quad f \in L^{2,e}(T,dv).$$

Spectral properties of the operators $H_{\gamma\lambda\mu}$

The perturbation operator $V_{\gamma\lambda\mu}$, $\gamma,\lambda,\mu \in R$ is of rank at most 3, therefore by the well known Weyl's theorem the essential spectrum $\sigma_{ess}(H_{\gamma\lambda\mu})$ of $H_{\gamma\lambda\mu}$ does not depend on $\gamma,\lambda,\mu \in R$ and coincides with the spectrum $\sigma(H_0)$ of H_0 (see), i.e.,

$$\sigma_{ess}(H_{\gamma\lambda\mu}) = \sigma(H_0) = [\min_{p\in T}\varepsilon(p), \max_{p\in T}\varepsilon(p)] = [0,2].$$

Auxilary statements.

For any $\gamma, \lambda, \mu \in R$, we introduce the Fredholm determinant $\Delta_{\gamma\lambda\mu}(z)$ associated to the Hamiltonian $H_{\gamma\lambda\mu}$ as

$$\Delta_{\gamma\lambda\mu}(z) = |1 + \gamma a(z) \quad \lambda b(z) \quad \mu c(z) \quad \gamma b(z) \quad 1 + \lambda d(z) \quad \mu e(z) \quad \gamma c(z) \quad \lambda e(z) \quad 1 + \mu f(z) \quad |,$$

where

$$a(z) := \int_T \frac{dv(t)}{\varepsilon(t) - z}, b(z) := \int_T \frac{costdv(t)}{\varepsilon(t) - z},$$

$$c(z) := \int_T \frac{cos2tdv(t)}{\varepsilon(t) - z}, d(z) := \int_T \frac{cos^2tdv(t)}{\varepsilon(t) - z},$$

$$e(z) := \int_T \frac{costcos2tdv(t)}{\varepsilon(t) - z}, f(z) := \int_T \frac{cos^2 2tdv(t)}{\varepsilon(t) - z}$$

are regular functions in $z \in R\backslash[0,2]$.

Lemma 1. For all $(\gamma, \lambda, \mu) \in R^3$ the number $z \in R\backslash[0,2]$ is an eigenvalue of the operator $H_{\gamma\lambda\mu}$ if and only if $\Delta_{\gamma\lambda\mu}(z) = 0$.

Proof. Lemma 1 can be proved using Fredholm determinants theory.

Lemma 2. The functions $a(\cdot)$,$b(\cdot)$, $c(\cdot)$, $d(\cdot), e(\cdot)$ and $f(\cdot)$ are regular in $R\backslash[0,2]$. Also, the following asymptotic relations are true for them:

$$a(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+,$$

$$d(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} - 1 + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+,$$

$$f(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} - 2 + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+.$$

$$b(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} - 1 + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+,$$

$$c(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} - 2 + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+, \ \ e(z) = -\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} - 2 + O(-z)^{\frac{1}{2}}, \ \ as\ z \to 2+.$$

Proof. The asymptotic relations of the above functions are calculated using residual theory.

Remark 3. Moreover, as the function under the integral sign are positive, monotonicity of the Lebesgue integral gives that the function $a(z)$ is negative too. Derivative of function $a(z)$ is negative, therefore this function monotonically increasing in the interval $(2, +\infty)$.

Lemma 4. The following asymptotics are true for the determinant $\Delta_{\gamma\lambda\mu}(z)$:

i. $\lim_{z \to \pm\infty} \Delta_{\gamma\lambda\mu}(z) = 1.$

ii. $\Delta_{\gamma\lambda\mu}(z) = C^+(\gamma,\lambda,\mu)\frac{1}{\sqrt{2}(-z)^{\frac{1}{2}}} + C_0^+(\gamma,\lambda,\mu) + O((-z)^{\frac{1}{2}}),\ as\ z \to 2+,$

where

$$C^+(\gamma,\lambda,\mu) = -\gamma - \lambda - \mu + \gamma\lambda + \lambda\mu + 2\gamma\mu - \gamma\lambda\mu.$$

and

$$C_0^+(\gamma,\lambda,\mu) = 1 - \lambda - 2\mu - \gamma\lambda - 2\lambda\mu - 4\gamma\mu - 2\gamma\lambda\mu.$$

Proof. The above asymptotics for the determinant $\Delta_{\gamma\lambda\mu}(z)$ can be derived by simply calculations and taking into account the asymptotics of the functions $a(\cdot), b(\cdot), c(\cdot), d(\cdot), e(\cdot)$ and $f(\cdot)$ in Lemma 3. □

Let we introduce the following set:

$$G_{3,+} = \{(\gamma,\lambda,\mu) \in R^3\colon C^+(\gamma,\lambda,\mu) < 0,\ A^+(\gamma,\lambda) > 0, \gamma > 1\},$$

where $A^+(\gamma,\lambda) = 1 - \lambda - 2\gamma + \gamma\lambda = 0.$

Theorem 5. Let $\gamma,\lambda,\mu \in R$. If $(\gamma,\lambda,\mu) \in G_{3,+}$the operator $H_{\gamma\lambda\mu}$ has three eigenvalues above the essential spectrum and has no eigenvalues below the essential spectrum.

Proof: Let we assume $(\gamma,\lambda,\mu) \in G_{3,+}$. It means $C^+(\gamma,\lambda,\mu) < 0,\ A^+(\gamma,\lambda) > 0, \gamma > 1$. According to positivity of γ and Remark 3. the function $\Delta_{\gamma 00}(z) = 1 + \gamma a(z)$ is continuous and monotone increasing in $(2,+\infty)$. Moreover

$$C^+(\gamma,0,0) = -\gamma < 0.$$

Lemma 4 yields that

$$\lim_{z\to 2+} \Delta_{\gamma 00}(z) = -\infty,\ \lim_{z\to+\infty}\Delta_{\gamma 00}(z) = 1.$$

Hence the function $\Delta_{\gamma 00}(z)$ has exactly one zero above the essential spectrum. Let we denote this zero by z_{11}. Since to monotonicity of the function we can find followings

$1 + \gamma a(z) < 0, if\ 2 < z < z_{11}$ and $1 + \gamma a(z) > 0, if\ z > z_{11}.$

Next we prove that the function $\Delta_{\gamma\lambda 0}(z)$ has two zeros greater than 2. To prove this fact we show that $\Delta_{\gamma\lambda 0}(z_{11}) < 0$ as follow:

$$\Delta_{\gamma\lambda 0}(z_{11}) = \big(1 + \gamma a(z_{11})\big)\big(1 + \lambda d(z_{11})\big) - \gamma\lambda b^2(z_{11}) = -\gamma\lambda b^2(z_{11}) < 0.$$

Due to $(\gamma,\lambda,\mu) \in G_{3,+}$ we can find that

$-\gamma - \lambda + \gamma\lambda = A^+(\gamma,\lambda) + (\gamma - 1) > 0.$

Equality $C^+(\gamma,\lambda,0) = -\gamma - \lambda + \gamma\lambda$ and Lemma 4. yields that

$$\lim_{z\to 2+} \Delta_{\gamma\lambda 0}(z) = +\infty,\ \Delta_{\gamma\lambda 0}(z_{11}) < 0,\ \lim_{z\to+\infty}\Delta_{\gamma\lambda 0}(z) = 1.$$

Hence the function $\Delta_{\gamma\lambda 0}(z)$ has two zeros above the essential spectrum. Let we denote these zeros by z_{21} and z_{22}.

In the below we prove that the function $\Delta_{\gamma\lambda\mu}(z)$ has three zeros greater than 3. Due to definition

$$\Delta_{\gamma\lambda\mu}(z) = \Delta_{\gamma\lambda 0}(z)\big(1 + \mu f(z)\big) + 2\gamma\lambda\mu b(z)c(z)e(z) - \gamma\mu(1 + \lambda d(z))c^2(z) - \lambda\mu(1 + \gamma a(z))e^2(z).$$

It is easy to check that

$$\Delta_{\gamma\lambda 0}(z_{21}) = (1 + \gamma a(z_{21}))(1 + \lambda d(z_{21})) - \gamma\lambda(b(z_{21}))^2 = 0.$$

Therefore

$\Delta_{\gamma\lambda\mu}(z_{21}) = \mu(\sqrt{\gamma}\sqrt{-(1+\lambda d(z_{21}))}c(z_{21}) + \sqrt{\lambda}\sqrt{-(1+\gamma a(z_{21}))}e(z_{21}))^2 > 0.$

Similarly can be shown

$$\Delta_{\gamma\lambda\mu}(z_{21}) = -\mu(\sqrt{\gamma}\sqrt{1+\lambda d(z_{21})}c(z_{21}) + \sqrt{\lambda}\sqrt{1+\gamma a(z_{21})}e(z_{21}))^2 < 0.$$

Due to above inequalities and Lemma 4 we can find following relations

$\lim_{z\to 2+} \Delta_{\gamma\lambda\mu}(z) = -\infty,\ \Delta_{\gamma\lambda\mu}(z_{21}) > 0,\ \Delta_{\gamma\lambda\mu}(z_{22}) < 0, \qquad \lim_{z\to+\infty}\Delta_{\gamma\lambda\mu}(z) = 1.$

Then the above relations yield the existence of zeros z_{31}, z_{32} and z_{33} of the function $\Delta_{\gamma\lambda\mu}(z)$ satisfying the following inequalities :

$$2 < z_{31} < z_{21} < z_{32} < z_{22} < z_{33}$$

Hence, if $(\gamma,\lambda,\mu) \in {G_3}^+$ then the function $\Delta_{\gamma\lambda\mu}(z)$ has three roots greater than 2. The Lemma 1 gives that the operator $H_{\gamma\lambda\mu}$ has three eigenvalues above the essential spectrum. Moreover, since to the perturbation operator $V_{\gamma\lambda\mu},\ \gamma,\lambda,\mu \in R$ is of rank at most 3, the operator $H_{\gamma\lambda\mu}$ has at most three eigenvalues outside the essential spectrum. Thus, the operator $H_{\gamma\lambda\mu}$ has no one eigenvalues below the essential spectrum. □

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